Order tracking using H_∞ estimator and polynomial approximation

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Abstract

In this paper we present the H_∞ estimator for discrete-time varying linear system combined with the polynomial approximation for order tracking of non-stationary signals. The proposed approach is applied to the gearbox diagnosis under variable speed condition. In this instance, it is well known that the occurrence of a fault on a gear tooth leads to the modulation of amplitude and phase of vibration signal orders. Our purpose is to estimate this unknown amplitude and phase modulation by tracking orders. In order to estimate these modulation signals, we model the vibration signal by using state variables. Then, we use the H_∞ criterion to minimize the worst possible amplification of the estimation error related to both the process and measurement noises. Such an approach doesn't require any assumption on the statistic properties of the noises unlike to the Kalman estimator. A numerical example is given in order to evaluate the performance of the H_∞ estimator regarding the conventional Kalman estimator.

Keywords: *Order tracking, $\boldsymbol{H}_\infty$ estimator, polynomial approximation and non-stationary conditions.*

1 Introduction

Order tracking using the state space approach is one of the tools widespread for the processing of non-stationary signals. The so-called the state space model is composed of two equation: the state equation and the measurement equation. The technique the most presented in this area is the Kalman estimator and more precisely the Vold_kalman estimator in the area of the mechanical systems diagnosis [1]. Vold et al. present the theoretical basis about this estimator in [2]. This kind of estimator suppose that the measurement noise and the process noise are centred, Gaussian and white with known statistics.

We can find in the literature many works on the Vold_kalman estimator for order tracking. M. Pan and Y. Lin have realized an interesting explorative study on the Vold_kalman estimator [3-4]. Behrouz and al. also applied this estimator to diagnose a bearing default and they have translated the state equation in term of second order autoregressive model [5]. These study have provided conclusive results. However, the unrealistic assumptions on the noises naturally limit the application of this estimator in real cases.

Therefore, we introduce in this paper the H_∞ estimator to evaluate the modulation of amplitude and phase of the orders. To track these modulation we first model the vibration signal using the state variables. Then these latter are modelled by a Taylor series. This method generalize that of Vold_kalman. With the H_∞ estimator we make no assumption on the noise statistics. But we assume only that the noises have finite energy. We find in the work of Shen and Deng [6] an introduction on the discrete H_∞ estimator.

This paper is structured as follow: section 2 presents the theoretical foundation about the H_∞ estimator and section 3 provides an example of simulation which validated our proposal.

2 Theoretical background

2.1 Problem formulation

The measurement signal we consider in this paper is modelled as

$$y(\mathrm{t}) = \sum_{i=1}^{M} A_i(\mathrm{t}) \cos\left(2\pi \int_0^t f_i(u)du + \phi_i(t)\right) + v(t) \quad \text{(eq. 1)}$$

Where A_i and ϕ_i are respectively the amplitude and the phase of the i^{th} order, v is the measurement noise which contains the unwanted part of the signal, $f_i = o_i f_r$ is the instantaneous frequency of the order of interest with f_r the reference frequency and o_i the value of the order i.

In the discrete form, (eq. 1) becomes:

$$y(k) = \sum_{i=1}^{M} A_i(k) \cos\big(\theta_i(k) + \phi_i(k)\big) + v(k) \quad \text{(eq. 2)}$$

$k = 0,2,\cdots,n-1.$

Where $\theta_i(k) = 2\pi \sum_{j=1}^{k} \frac{f_i(j)}{f_s}$ is the angular displacement and f_s is the sampling frequency.

Our objective is to estimate the amplitude and the phase of some specific orders of interest using the H_∞ estimation approach. For this, we formulate the problem in term of estimation of the state variables. And we must keep in mind that these amplitude and phase are features of the fault on the gear teeth.

2.2 State space modelling

Let us consider the formula established in the (eq. 2).The purpose here is to build the measurement and the state equation.

Linearizing the (eq. 2) we get:

$$y(k) = \sum_{i=1}^{M} [\cos(\theta_i(k)) \quad -\sin(\theta_i(k))] \begin{bmatrix} a_{i,c}(k) \\ a_{i,s}(k) \end{bmatrix} + v(k) \quad \text{(eq. 3)}$$

Where $a_{i,c} = A_i cos\phi_i$ and $a_{i,s} = A_i sin\phi_i$. Let put $a_i(k) = \begin{bmatrix} a_{i,c}(k) \\ a_{i,s}(k) \end{bmatrix}$ and

$B_i(k) = [\cos(\theta_i(k)) \quad -\sin(\theta_i(k))]$.

The amplitudes $a_{i,c}$ and $a_{i,s}$ are unknown. For estimating them we model these amplitudes by a polynomial approximation as follows:

$$a_{i,c}(k) = \sum_{q=0}^{N} \alpha_{i,c}^q(k) t^q(k) \quad \text{(eq. 4)}$$

$$a_{i,s}(k) = \sum_{q=0}^{N} \alpha_{i,s}^q(k) t^q(k),\ i = 1,2,\dots,M \quad \text{(eq. 5)}$$

and the coefficients of the polynomial by a random walk process such as:

$$\alpha_{i,c}^q(k+1) = \alpha_{i,c}^q(k) + w_{i,c}(k) \quad \text{(eq. 6)}$$

$$\alpha_{i,s}^q(k+1) = \alpha_{i,s}^q(k) + w_{i,s}(k) \quad \text{(eq. 7)}$$

where $w_{i,.}$ is a random signal. With those new variables (eq. 3) can be rewritten as:

$$y(k) = T(k)B(k)x(k) + v(k) \quad \text{(eq. 8)}$$

With $(k) = [1 \quad t(k) \quad \cdots \quad t^N(k)]$, $B(k) = [B_1(k) \quad B_2(k) \quad \cdots \quad B_M(k)]$ and

$x(k) = [x_1(k) \quad x_2(k) \quad \cdots \quad x_M(k)]^T$ such as $x_i(k) = \begin{bmatrix} x_{i,c} \\ x_{i,s} \end{bmatrix}$ with $x_{i,.} = [\alpha_{i,.}^0 \quad \alpha_{i,.}^1 \quad \cdots \quad \alpha_{i,.}^N]^T$.

Note that A^T is the transpose of the matrix A.

Assuming that the measurement matrix is $H(k) = T(k)B(k)$, we finally obtain the measurement equation just below:

$$y(k) = H(k)x(k) + v(k) \quad \text{(eq. 9)}$$

Where v is the measurement noise with a covariance matrix V.

Then the state equation are:

$$x(k+1) = Fx(k) + w(k) \quad \text{(eq. 10)}$$

Where $F = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$ and $w(k) = [w_{1,c} \quad w_{1,s} \quad w_{2,c} \quad w_{2,s} \quad \cdots \quad \cdots \quad w_{M,c} \quad w_{M,s}]^T$ is the process noise with a covariance matrix W.

2.3 Discrete H_∞ estimator design

From the (eq. 9) and (eq. 10) we get the following state space model:

$$\begin{cases} x_{k+1} = & Fx_k + Bw_k \\ y_{k+1} = & H_k x_k + v_k \end{cases} \quad \text{(eq. 11)}$$

Let us note $e_k = x_k - \hat{x}_k$ the estimation error where $\hat{x}_k$ is the estimate of x_k and $E\{.\}$ will stand for the expectation value.

Several facts may be used against the Kalman estimator although it is an attractive and powerful tool to estimate x_k:

1. The Kalman estimator minimizes $E\{e_k e_k^T\}$ while the user may be interested in minimizing the worst-case error.
2. The Kalman estimator assumes that the noises are zero-mean with Gaussian distribution.
3. The Kalman estimator assumes also that $E\{v_k v_k^T\}$ and $E\{w_k w_k^T\}$ are known.

These limitations have led to the statement of the H_∞ estimation problem. Several formulations exist in the literature. The H_∞ estimator solution that we present here is originally developed by Ravi Banar [7] and further explored by Shen and Deng [6]. These pioneer define the following cost function:

$$J = \frac{\sum_{k=0}^{n-1} \|x_k - \hat{x}_k\|_Q^2}{\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 + \sum_{k=0}^{n-1}\left(\|w_k\|_{W^{-1}}^2 + \|v_k\|_{V^{-1}}^2\right)} \quad \text{(eq. 12)}$$

Where $\hat{x}_0$ is an estimate of x_0, $Q > 0$, $P_0 > 0$, $W > 0$ and $V > 0$ are the weighting matrices and are left to the choice of the designers and depend on the performance requirements. The notation $\|x_k\|_Q^2$ defines the weighted $Q - L_2$ norm, i.e, $\|x_k\|_Q^2 = x_k^T Q x_k$.

Problem statement [8]: Given the scalar $\gamma > 0$, find estimation strategy that achieve

$$\sup J < 1/\gamma \quad \text{(eq. 13)}$$

Where "sup" is the supremum value and γ is the desired level of noise attenuation.

The H_∞ estimation problem consists on the minimization of the worst possible amplification of the estimation error. This can be interpreted as a "minmax" problem since we are searching to minimize the estimation error and to maximize the exogenous disturbances (w_k and v_k) and the error of initialization ($x_0 - \hat{x}_0$).

Remember that unlike the Wiener/Kalman estimator, the H_∞ estimator deals with deterministic noises and no *a priori* information on their statistic properties are required. The solution of the H_∞ estimation problem is given in the theorem below from [6].

Theorem: Let $\gamma > 0$ be a prescribed level of noise attenuation. Then, there exists a H_∞ estimator for x_k if and only if there exists a stabilizing symmetric solution $P_k > 0$ to the following discrete-time Riccati equation:

$$P_{k+1} = FP_k(I - \gamma QP_k + H_k^T V^{-1} H_k P_k)^{-1} F^T + \mathrm{B}WB^T \quad \text{(eq. 14)}$$

Then the H_∞ estimator gives the estimate $\hat{x}_k$ of x_k such as:

$$\hat{x}_{k+1} = F\hat{x}_k + K_k(y_k - H_k\hat{x}_k), \hat{x}_0 = x_0 \quad \text{(eq. 15)}$$

K_k is the gain of the H_∞ estimator and is given by:

$$K_k = FP_k(I - \gamma QP_k + H_k^T V^{-1} H_k P_k)^{-1} H_k^T V^{-1} \quad \text{(eq. 16)}$$

Another way to solve the Riccati equation (eq. 14) is presented by Yaesh and Shaked [9]. The method is given as follows:

1. Form the Hamiltonian

$$Z = \begin{bmatrix} F^{-T} & F^{-T}[H^T R^{-1} H - \gamma I] \\ BQB^T F^{-T} & F + BQB^T F^{-T}[H^T R^{-1} H - \gamma I] \end{bmatrix} \in \mathbb{R}^{2n*2n} \quad \text{(eq. 17)}$$

Where n is x dimension.

2. Find the eigenvectors of Z corresponding to the eigenvalues $\mathcal{E}_i (i = 1, \cdots, n)$ outside the unit circle
3. Form the matrix of the corresponding eigenvectors denoted by

$$(\mathcal{E}_1\ \mathcal{E}_2\ \cdots\ \mathcal{E}_n) \equiv \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \quad \text{(eq.18)}$$

Where $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}^{n*n}$.

4. Compute $P = \mathcal{X}_2 \mathcal{X}_1^{-1}$.

Note that more γ gets smaller, more the problem is easier to solve. When γ tends to γ_{opt} (the optimal value of γ) the eigenvalues of P tend to infinity and therefore $\mathcal{X}_1$ is close to a singular matrix. Shaked and Theodor [10] investigate the behaviour of the optimal H_∞ estimator when γ tends to γ_{opt}. They showed that when we are close to the optimum value for γ, there exists at least one or more unbounded eigenvalues.

In the special case, where $\gamma \rightarrow 0$, the H_∞ estimator reduces to a Kalman estimator.

3 Numerical implementation

In this section, we use an example to illustrate the performances of H_∞ estimation approach. The generated signal (see Fig. 1) is described by the following equation.

$$y(t) = \sum_{i=1}^{3} A_i(t) \cos(2\pi o_i \int_0^t f_r(u) du) + v(t) \quad \text{(eq. 19)}$$

Where f_r is the instantaneous frequency linearly increasing from 0 to 50 Hz in 5 $secondes$, o_i contains the order's number and v is the measurement noise.

The signal is composed of three orders presented in the table 1. Figure 2 displays the rpm-frequency spectrum using the conventional windowing Fourier transform that characterizes three orders.

Order number	1	4	9
Amplitude	Linearly increasing From 0 to 10	Linearly increasing From 3 to 13	Fixed at 10

Tab. 1. The synthetic signal's amplitude of orders

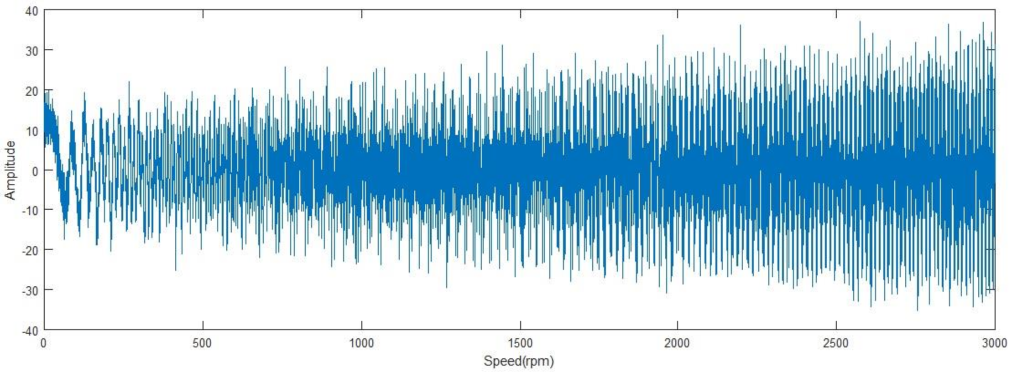

Fig. 1. Synthetic signal

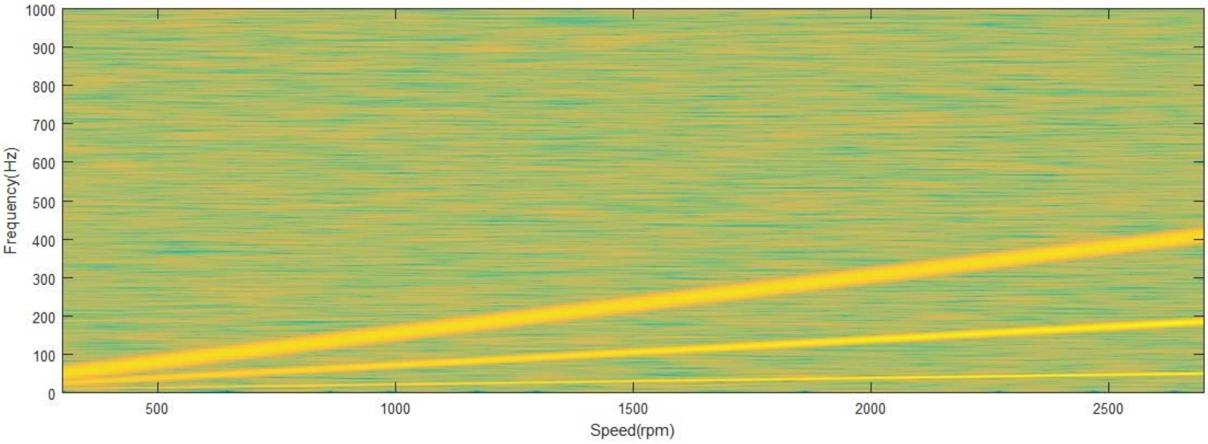

Fig. 2. Illustration of rpm-frequency spectrum

The results presented below have been got using a Monte-Carlo simulation based on 400 iterations. The parameters of the estimator have been taken as follows:

- the covariance of the process noise $W = 10^{-9}$,
- the covariance of the measurement noise $V = 10^{-3}$,
- the initial covariance error $P_0 = 10^{-3}$,
- the level of the noise attenuation $\gamma = \gamma_{opt} = 10^{0.178}$.

γ_{opt} is equal to γ to the greatest value that guarantees the stability of the matrix P. This stability is reached, according to Yaesh and Shaked [9], when the P's eigenvalues are bounded in the unit circle. As plotted in the figure 3, we achieve this stability for $\gamma = 10^{0.178}$. Beyond this value there exists at least one or more eigenvalues that are outside the unit circle.

The measurement noise is modelled by a noise of Poisson as mentioned in [11].The Kalman estimator algorithm presented by Dan Simon [12] and the H_∞ estimator have been applied to the generated vibration signal. The performance of both estimators is measured in term of signal to noise ratio. The table 2 gives the performance got for the two estimators. In both cases the H_∞ estimator provides a better result than the Kalman estimator. The SNR_{out} value is the signal to noise ratio calculated by

$$SNR_{out} = 10 * \log_{10} \frac{\sum_{k=1}^{N} y_k^2}{\sum_{k=1}^{N} (y_k - \hat{y}_k)\text{^}2} \qquad \text{(eq. 18)}$$

Where N is the number of samples, y_k is the noiseless signal at times k and $\hat{y}_k$ is the estimated or filtered signal. The criterion of comparison is improved by about $0.7\,dB$ using the H_∞ estimator. Therefore the H_∞ estimator is a good alternative to deal with real situation where the noises are not really Gaussian.

SNR_{in}	Estimation algorithm	SNR_{out}	
		White Gaussian noise	Poisson noise
$5\ dB$	Kalman H_∞	29.9771 30.7015	23.3424 23.7324
$10\ dB$	Kalman H_∞	39.9468 40.6510	33.2658 33.8200
$15\ dB$	Kalman H_∞	49.2331 49.9383	43.0675 44.0972

Tab. 2. Performance comparison between Kalman and H_∞ filtering

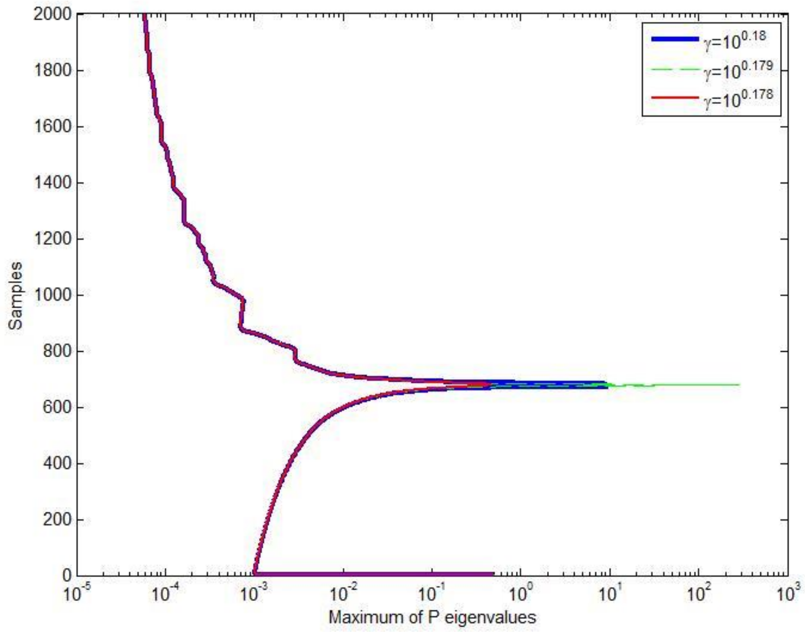

Fig. 3. Maximum of the eigenvalues of the covariance matrix error

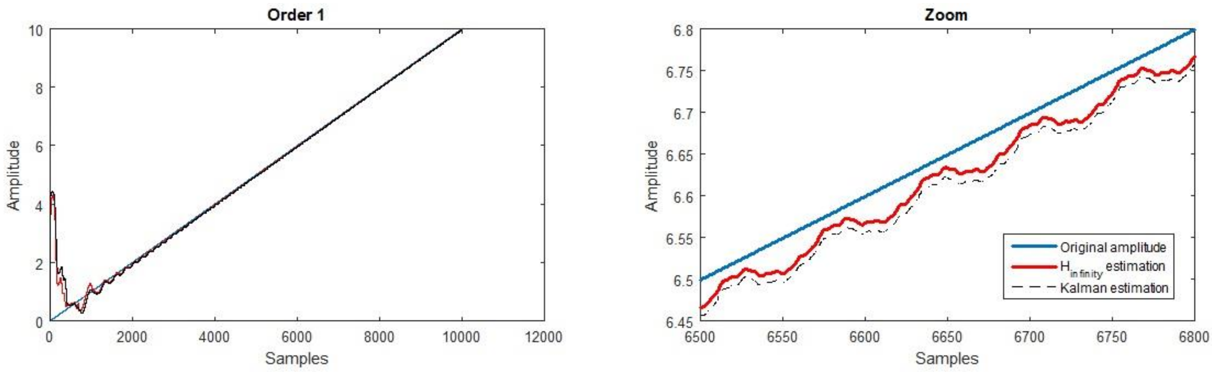

Fig. 4. Amplitude of order 1 estimated using the H_∞ and the Kalman estimator

Figure 4 to figure 6 show the effectiveness of the H_∞ estimator for order tracking in non-stationary signal processing. We see in this last figure that the estimated we got by the H_∞ estimation is more close to the original amplitude than the Kalman estimation.

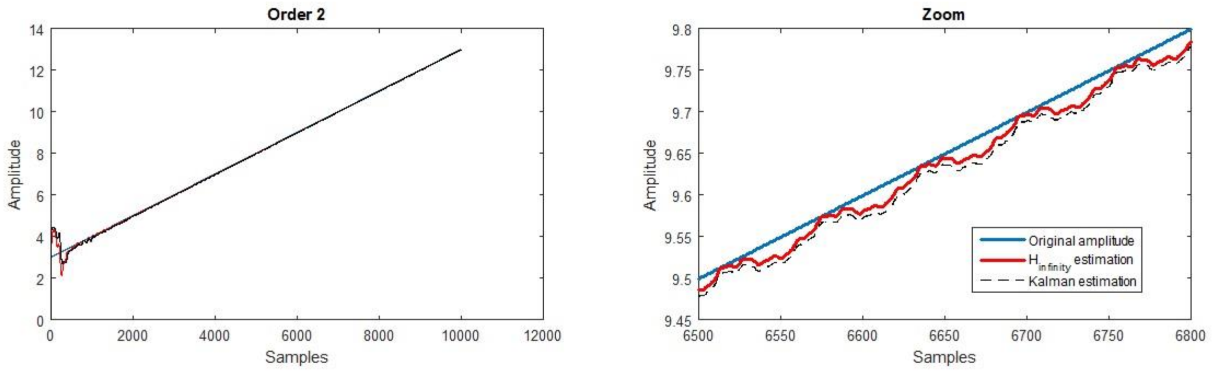

Fig. 5. Amplitude of order 2 estimated using the H_∞ and the Kalman estimator

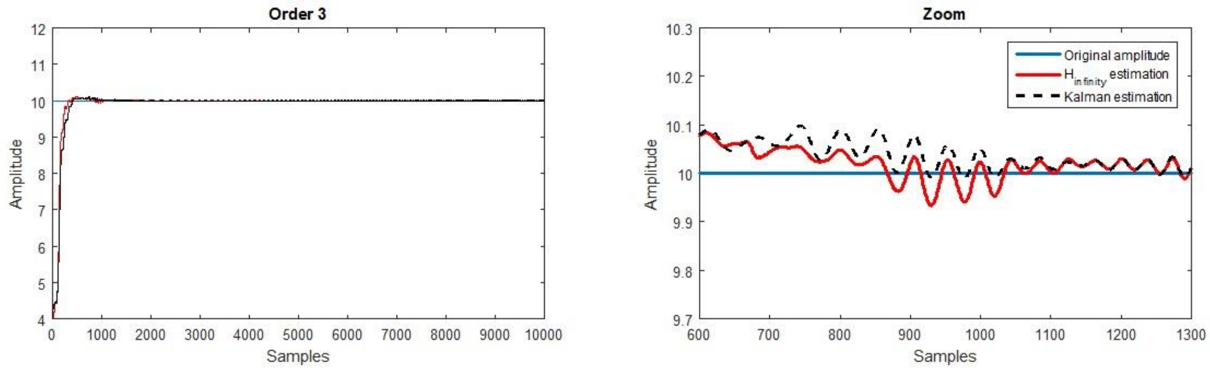

Fig. 6. Amplitude of order 3 estimated using the H_∞ and the Kalman estimator

4 Conclusion

Through this paper we developed a method to estimate order's amplitude based on the H_∞ estimation in non-stationary operations. This method uses the information about the instantaneous frequency of the signal and make no assumption on the noises statistics. It take advantage on the classical Kalman estimation and it can be consider as an extension of this last one. Since the estimator is designed to minimize the worst case-disturbances, the H_∞ estimation approach is more robust to process any kind of noisy signal. The application of this method in real-life data will concern our future research.

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