NEW COMBINED WAVELET METHODOLOGY FOR VIBRATION-BASED DAMAGE DETECTION

DETECTION DE DEFAUT PAR UNE NOUVELLE METHODOLOGIE BASEE SUR LES ONDELETTES COMBINEES

Roger SERRA[1,*], Lautaro LOPEZ[1,2]

[1] INSA Centre Val de Loire, Laboratoire de Mécanique et Rhéologie, 3 rue de la Chocolaterie, 41000 Blois, France

** corresponding author : roger.serra@insa-cvl.fr*

[2]Universidad Nacional de Rosario (UNR), Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Rosario, Argentina

Abstract

Different approaches on the detection of damages based on dynamic measurement of structures have appeared in the last decades. They were based, amongst others, on changes in natural frequencies, modal curvatures, strain energy or flexibility. Wavelet analysis have been used to detect the abnormalities on modal shapes induced by damages. However the majority of previous work was made with non-corrupted by noise signals. Moreover, the damage influence for each mode shape was studied separately. This paper proposes a new methodology to cope with noisy signals, while at the same time, measure the useful information each mode shapes has. The proposed methodology will be compared with ones the most famous and wide-studied methods in the bibliography. To evaluate the performance of each method, their capacity to detect and localize damage will be analyzed in different cases. The comparison will be done by simulating a cantilever steel beam as a numerical case. The proposed methodology proved to outperform classical methods in terms of noisy signals. The futur part of this work with experimental data will validate the practicability of the proposed methodology for real complex cases.

Keywords: damage assessment / modal analysis /wavelet transform/damage indicator/ cantilever beam.

Résumé

Différentes approches sur la détection des endommagements basées sur des mesures dynamiques des structures sont apparues au cours des dernières décennies. Elles étaient fondées, entre autres, sur les changements dans les fréquences naturelles, courbures modales, énergie de déformation ou de flexibilité. L'analyse par ondelettes a été utilisée pour détecter les anomalies sur les formes modales produites par les dommages. Toutefois, la majorité des travaux précédents a été faite avec des signaux sans bruit.De plus, l'influence de dommages pour chaque forme modale a été considérée séparément.Cet article propose une nouvelle méthodologie pour aborder des signaux bruités afin de mesurer l'information utile de chaque forme modal. La méthodologie proposée sera comparée aux méthodes les plus étudiés dans la littérature. Pour évaluer la performance de chaque méthode, leur capacité à détecter et localiser les dommages seront analysés dans divers cas. La comparaison sera effectuée sur unesimulation numérique d'une poutre encastrée-libre en acier. La méthodologie proposée a prouvé son efficacité supérieure aux méthodes classiques notamment en présence de forts bruits. La prochaine partie de ce travail validera la faisabilité de la méthode proposée pour les cas réelsavec des données expérimentales.

Mots clés: évaluation des dommages / analyse modale / transformée en ondelettes / indicateur d'endommagement / poutre encastrée-libre

Nomenclature

L	= Length of the beam (m)	$CWT(a,b)$	= Wavelet transform coefficients

W	= Width of the beam (m)	n	= Number of vanishing moments
H	= Depth of the beam (m)	θ	= Smoothing function
E	= Young's modulus of material ($N.m^{-2}$)	j	= Discrete translation parameter
ρ	= Density of material ($kg.m^{-3}$)	k	= Discrete dilation parameter
v	= Poisson ratio of material	$d_{j,k}$	= Detailed coefficients
L_c	= Crack location from clamped side (m)	$a_{j,k}$	= Approximated coefficients
W_c	= Crack width (m)	$\varphi(x)$	= Scaling function
H_c	= Crack depth (m)	$\{h_n\}$	= Low-pass filter response
$\psi(x)$	= Mother wavelet	$\{g_n\}$	= High-pass filter response
a	= Continuous translation parameter	$f(x)$	= Analyzed function
b	= Continuous dilation parameter		

1 Introduction

Damages such as cracks and pores are inevitable in aerospace, aeronautical, mechanical and civil engineering structures during their service life. The study of damages in structures is an important prospect to evaluate in order to ensure their safety and prevent sudden collapses. Some structures, such as large bridges, are required to be continuously evaluated to detect possible damages and repair them as soon as possible to never stop their service. Moreover, such damage identifications have significant life-safety implications.

Structural Health Monitoring is an efficient way to evaluate the state of a structure. Consequently it guarantees safety and efficiency of structures throughout their life. This process involves the measurement of a structure over time, the extraction of damage-sensitive features from the measurements and their analysis to determine the current state of the system. Assessment methods should be able to diagnose damages at an early stage, be non-destructive and low-cost. Many methods were proposed along history to identify problems in structures. Visual inspection was one of the first to appear, however detecting cracks is not always possible with this method since some zones of the structure might be restricted to sight. Nowadays, cracks are generally being detected by non-destructive testing methods (ultrasonic testing, radiographic, X-ray, eddy-current, ultrasonic, etc.). Vibration-based Structural Health Monitoring is one based on the fact that the loss of stiffness due to damage affects the dynamic response of the structure. It is mainly attractive because of its ability to monitor and detect damage from a global testing of the structure. Thus, it does not require accessibility and measuring of potentially damaged areas.

In this paper, a new approach for damage identification based on Wavelet transform is proposed in which the drawback of Cao and Quiao [1] methodology are dealt with the modal treatment presented in Solís et al [2]. The result is a noise-robustness approach that makes use of modal data in a more efficient way. First, extracted or simulated modal shapes of a damaged beam refined in order to reduce random noise and interferences. The difference between undamaged modal shapes and the refined ones are weighted depending on the changes in corresponding natural frequencies. Finally, the addition of the coefficients of CWT for each difference in modal shapes is used to localize damages.

Wavelet background is firstly presented in this article. Then the proposed methodology is described. Several numerical examples are provided to be able to compare some of the classical methods with the proposed one. The numerical data, which involves the calculation of the first few natural frequencies and modal shapes from a cracked Euler-Bernoulli beam, are obtained with COMSOL. Nine cases with different crack depth and positions are studied. The influence of noise in modal shapes is also presented. To sum up, conclusions are drawn.

2 Literature review

The first approach for using modal analysis for damage detection was based on the fact that changes in the properties of structures modify the fundamental frequencies of vibration. Cawley and Adams [3] were one of the first to develop a method based on the shift of more than one frequency that could yield to the location of the damage. Salawu [4] presented a review on the use of natural frequency changes for damage detection. Recently, Gautier et al.[5] have studied the performance of Subspace Fitting Methods as well as hybrid methods (GA, PSO, etc.) to detect and locate damages in a structure.

The change in natural frequencies is not considerable when the damage is small, which makes preventive maintenance difficult to carry out. As a result, damage indicators based on vibrational modal shapes were introduced. They were mainly attractive because the change in stiffness provoked by the damage is reflected locally in the modal shapes. Allemagne and Brown [6] presented probably the first use of mode shape information for damage detection in a structure. Modal Assurance Criterion was used to determine if a body flap was damaged or not. In Kim et al. [7] a comparison between frequency-based damage detection methods and modal shape-based detection methods was carried out. The authors formulated algorithms for bots methods to locate damage and estimate the size of the simulated crack.They concluded that changes in modal shapes are much more sensitive compared to changes in natural frequencies.Coordinate Modal Assurance Criterion was introduced by Lieven and Ewins [8], which is conceptually similar to MAC but it tries to solve the problem of location of the damage. Pandey et al. [9] introduced the concept of mode shape for damage localization. In their study, both a cantilever and a simply supported beam model were used to demonstrate the effectiveness of changes in modal curvature as damage indicator to detect and localize damage. Wahab and De Roeck [10] found that the modal curvature of lower vibration modes was in general more accurate than those of the higher vibration modes. The advantage of the curvature mode shape over the displacement mode shape in identifying the location of damage was demonstrated by Salawu and Wiliams [11]. Based on the curvature method, Stubbs et al.[12] proposed the Damage Index Method, which is the comparison between square curvatures of the structure before and after the damage. An approach based on the use of strain energy to detect and localize damage was presented by Sazonov and Klinkhachorn[13].The main drawback of the mentioned methods is the needto calculate derivatives numerically. The differential operator entails numerical error which might be bounded but not negligible. Pandey and Biswas [14]proposed the use of changes in the dynamically measured flexibility matrix to detect and localize damage. They showed that the flexibility matrix of a structure can be easily and accurately estimated from a few low frequency vibration modes of the structure.

Another important approach for damage detection is based on Wavelet Transform. The Wavelet Transform is a mathematical tool that decomposes a signal into a localized time (or scale) and frequency (or scale) domain, as Daubechies [15] explained. Wavelets are purposefully crafted to have specific properties that make them useful for signal processing. As a matter of fact, Wavelet Analysis is capable of revealing "hidden" aspects of the input signal, which in case of SHM are provoked by damages. The main advantages of Wavelet Analysis is the ability to perform local analysis of signals, its noise robustness and the possibility to assess the state of a structure without the baseline undamaged state of the structure. The first publication considered to mention Wavelet Analysis as damage indicator is Surace and Ruotolo [16] in 1994. Liew and Wang [17] proposed a damage detection method based on the sudden changes in the variation of Wavelet Transform coefficients. Hong et al.[18] and Douka et al.[19] used Lipschitz exponent to localize and estimate the size of a crack. They came to the conclusion that the size of the crack can easily be determined by the coefficients of wavelet transform. Their results were confirmed both numerically and experimentally. Chen et al.[20] used Lipschitz exponent for the same reason and proved its efficiency with different types of excitation. They also analyzed the influence of damage location and mother wavelets as well as noise in signals. Wu and Wang [21] used high resolution laser with CWT to measure the deflection profile of a cracked cantilever beam to locate its damage.Solís et al.[2] presented a method based on the addition of the coefficient of CWT for different mode shapes weighted depended on the changes in natural frequencies for each one. However, in their laboratory work they had to apply several impacts in order to obtain 65 measurement points and reduce the experimental noise. This methodology may result expensive and not practical in real cases scenarios.

Discrete Wavelet Transform takes into account much less situations to calculate coefficients but the analysis will be more efficient in terms of calculation work. Naldi and Venini [22] in 1997 and Lu and Hsu [23] in 1999 were one of the first to use Discrete Wavelet Analysis to detect and localize damage.

They analyzed a 1-D structure and flexible strings respectively and came to the conclusion that this method is capable of correctly assess the state of a structure using its dynamics characteristics. Even though DWT is widely used in structural and machine health monitoring, it is a non-redundant decomposition analysis, as Mallat [24] explains. This provokes DWT to be shift-variant. In a shift-invariant system, the contemporaneous response of the output variable to a given value of the input variable does not depend on when it occurs. This feature is very important for monitoring systems. To cope with this issue, Stationary Wavelet Transform (SWT) was introduced which makes the wavelet transform redundant, A comparison between DWT and SWT is shown in Zhong and Oyadiji [25] where the authors concluded that SWT is much more suitable for damage detection than DWT.

The majority of the studies were numerical On the other hand, inthe experimental ones like Rucka and Wilde [26], great amount of sensors had to be used to deal with random interferences in the measured data and noise.A technique called Integrated Wavelet Analysis, which combines the advantages of SWT and CWT, was proposed by Cao and Quiao [1].Their method implements SWT to refine the modal shapes and then look for damage applying CWT to the refined modal shape. Masoumi and Ashory [27]made numerical and experimental studies on this method to localize cracks. However, they only took into account the first modal shape in their calculations. It is well-known that only one mode shape is not enough for damage detection since the sensitivity of the mode depends on the location of the damage

3 Wavelet background

Both the Fourier and Wavelet Transform measure *similarity* between a signal and an analyzingfunction. The two transforms differ in their choice of analyzing function. In the Fourier transform, the analyzing functions are complex exponentials e^{jwx} where x is the independent variable space. In Wavelet Transform, the analyzing function is a wavelet ψ. Wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Daubechies [15] and Mallat [28] developed the principal facts on Wavelet Transform Theory.

A mother wavelet $\psi(x)$ can be defined as an oscillating function with zero average value:

$$\int_{-\infty}^{+\infty} \psi(x)dx = 0 \quad (1)$$

The function $\psi(x)$is used to create a family of wavelets$\psi_{a,b}(x)$:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \quad (2)$$

Where real numbers a and b are dilation and translation parameters, respectively.

3.1 Continuous Wavelet Analysis (CWT)

The CWT compares the input signal to shifted and compressed or stretched versions of a mother wavelet. Stretching or compressing a function corresponds to the notion of *scale*. By comparing the signal to the wavelet at various scales and positions, function of two variables is obtained.

For a scale parameter $a > 0$ and position b, the CWT is defined as:

$$CWT_f(a,b) = \int_{-\infty}^{+\infty} f(x)\frac{1}{\sqrt{a}}\psi^*\left(\frac{x-b}{a}\right)dx \quad (3)$$

ψ^*denotes the conjugate of the mother wavelet. By varying the values of the scale parameter$a$, and the position parameter b, CWT coefficients $CWT_f(a,b)$are obtained.Regarding the space-scale view of signals, it is worth analyzing the ability of wavelet to react to subtle changes or discontinuities in signals. An important feature of a wavelet is its number of vanishing moments. If a wavelet has n vanishing moments, then:

$$\int_{-\infty}^{+\infty} x^k\psi(x)dx = 0\, fork = 0 \ldots n-1 \quad (4)$$

For any polynomial of smaller order than the number of vanishing moments, the wavelet will give null values. The number of vanishing moments indicates how sensitive the wavelet is to low order signals, and can be chosen to take into account only certain components of the signal.

It can be mathematically demonstrated (Mallat[28]) that for a wavelet with *n* vanishing moments and a fast decay there exist a function $\theta(x)$ called *smoothing function* defined as follows:

$$\psi(\mathrm{x}) = \frac{d^n \theta(\mathrm{x})}{dx^n} \,, \int_{-\infty}^{+\infty} \theta(t)\mathrm{dt} \neq 0 \qquad (5)$$

Therefore, a wavelet with n vanishing moments can be rewritten as the *n*th order derivative of the smoothing function $\theta(\mathrm{x})$.Continuous Wavelet Transform Wavelet coefficients can be expressed as:

$$\begin{aligned} CWT_f(a,b) = \int_{-\infty}^{+\infty} f(x)\, \psi^*\left(\frac{x-b}{a}\right) dx &= \frac{a^n}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \frac{d^n\theta}{dx^n}\left(\frac{x-b}{a}\right) dx \\ &= \frac{a^n}{\sqrt{a}} \frac{d^n}{dx^n} \int_{-\infty}^{+\infty} f(x)\, \theta\left(\frac{x-b}{a}\right) dx \end{aligned} \qquad (6)$$

The wavelet transform depends on the *n*th derivative of the signal function. If a signal has a singularity at a certain point *u i.e.* it is not differentiable at *u,* then the coefficients will have large values. When the scale is large, the function removes small signal fluctuation an only detects large variations.

The results of the transform are wavelet coefficients that show how well a wavelet function correlates with the signal analyzed. A sharp transition in $f(x)$creates wavelet coefficients with large amplitude. In the analyzed case, the transition in $f(x)$ will be caused by a crack and this precisely is the proposed identification method.

3.2 Discrete and Stationary Wavelet Analysis (DWT & SWT)

Discrete Wavelet Analysis will result in less amount of information since it is going to be performed only on a discrete grid of parameters of dilatation and translation. An arbitrary signal $f(x)$ can be written using an orthonormal wavelet basis:

$$f(x) = \sum_j \sum_k \mathrm{d}_{j,k}\, \psi_{j,k}(x) \qquad (7)$$

Where the coefficients are given by:

$$\mathrm{d}_{j,k} = \int_{-\infty}^{+\infty} f(x)\, \psi_{j,k}(x)\, dx \qquad (8)$$

The orthonormal basis functions are generated by dilating and translating an orthonormal mother wavelet$\psi(\mathrm{x})$:

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-\frac{j}{2}}x - k) \quad (9)$$

Where j and k are the dilation and translation parameters, respectively. The contribution of the signal at level j is given by:

$$d_j(x) = \sum_n d_{j,k}\, \psi_{j,k}(x)\, dx \quad (10)$$

This coefficients are called *detailed coefficients* and provide information on the space behavior of the signal at different scales. Mallat [28] developed a computationally efficient method to calculate Equations (7) and (8) from finite signals, which is called as multiresolution analysis (MRA).A multiresolution decomposition of a signal is based on successive decomposition into approximation and detailed coefficients. In order to implement fast wavelet transform to perform MRA,DWT and SWT make use of mother wavelets as well as scaling functions and approximate coefficients, which are related to the mother wavelet. The scaling function $\varphi(x)$ is translated and dilated as well:

$$\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-\frac{j}{2}}x - k) \quad (11)$$

Both functions satisfy the two-scale equation:

$$2^{-j/2}\psi\left(\frac{x}{2} - k\right) = \sum_{n=-\infty}^{+\infty} g(n-2k)\,\varphi(x-n) \quad (12)$$

$$2^{-j/2}\varphi\left(\frac{x}{2} - k\right) = \sum_{n=-\infty}^{+\infty} h(n-2k)\,\varphi(x-n) \quad (13)$$

Where $\{h_n\}$ and $\{g_n\}$ are the impulse responses of low-pass and high-pass filters, respectively.

The approximate and detailed coefficients at level j can be calculated from the approximate coefficients at level $j+1$:

$$a_{j,k} = \sum_n h\,(n-2k)\,a_{j+1,n} \quad (14)$$

$$d_{j,k} = \sum_n g\,(n-2k)\,a_{j+1,n} \quad (15)$$

Concerning the definitions already done, a signal $f(x)$can be decomposed up to the Jth level as:

$$f(x) = \sum_{j=1}^{J} (\sum_{k=1}^{N} d_{j,k}(k)\, \psi_{j,k}(x)) + \sum_{k=1}^{N} a_{j,k}(k)\, \varphi_{j,k}(x) \quad (16)$$

Approximation coefficients have information about the identity of the signal, while detailed coefficients give information about local discontinuities. DWT is able to split a multicomponent signal into a group of component signals. Measured signals are typically a mixture of noise components with damage-related information. In this regard, the extraction of damage-related component of the signal would be useful.

As it was stated, DWT is a non-redundant decompositionwhich is a serious drawback for damage detection. The non-redundancy ofDWT is caused by the decimation operation at each level of decomposition, which modifies the number of sampling points for each level. This drawback can be solved by using SWT, which is similar to DWT except that the number of wavelet coefficients at each level remains equal to the number of sampling points while when using DWT the reduction of the number of coefficients is $n/2^j$. SWT doubles the number of input samples at each iteration. Since SWT also has the capability of decomposing a multicomponent signal, it is going to be used to refine the modal shapes and obtain the damage-related information.

3.3 Wavelet families and properties

There are many different admissible mother wavelets that can be used in wavelet analysis. The election of the mother wavelet depends on the signal which is going to be tested. It is worth remembering that not only the value of the scale and position affects the value of wavelet coefficients but also the choice of the wavelet. This choice should be based on a mathematical analysis of the nature of the signal. However this analysis is not always clear and the wavelet family is usually determined by trial and error. The lack of theoretical investigation on the choice of the optimal mother wavelet family is a problem since the effectiveness of the application is directly implicated to the wavelet family.

Some authors, like Ovanesova and Suárez[29] and Cao and Quiao [1] have briefly studied the most important wavelet properties, which are:

- Regularity: if *r* is an integer and a function is *r*-times continuously differentiable, then the regularity is *r.* The regularity is useful to reach a regular spatial-scale representation of wavelet coefficients.
- Size of support: is the smallest space-set (or time-set) outside of which function is equal to zero.
- Symmetrythe symmetry avoids the phase distortion thusconserves the spatial localization of wavelet coefficients. In case of a cracked beam, the failure is supposed to cause a discontinuity in mode shapes, thus it is required that the mother wavelet had this property.

In the case of Stationary Wavelet Analysis, the mother wavelet should also have an explicit scaling function. After this exigency, orthogonal and biorthogonal wavelets still remain.

Ovanesova and Suárez [29], Cao and Qiao[1], and Prasad et al [30] took the characteristics previously described and concluded that reverse biorthogonal wavelets are the most appropriated since they enjoy merits as orthogonality, short size of support and symmetry.

As mentioned above, the number of vanishing moments is critical for spatial detection of singularities in wavelet analysis. Cao and Qiao [1] demonstrated that even number of vanishing moments are more efficient in showing signal discontinuities.

It was stated before that a wavelet with *n* vanishing moments is orthogonal to polynomials up to degree *n-1*. The number of vanishing moments indicates how sensitive the wavelet is to low order signals, and can be chosen to take into account only certain components of the signal. Therefore, if a wavelet with 2 vanishing modes is used, the CWT coefficients of mode shapes give information about the change in the second derivatives of mode shapes (modal curvature). It is known that changes in curvatures is a damage indicator. Also, 2 vanishing moments is considered the minimum value for crack detection (X.Jiang et al.[31]).

$$n \geq 2 \quad (36)$$

Another way to cope with the number of vanishing moments is identifying that the first modal shapes can be approximated using polynomials up to degree not less than 2 but no more than 4. Thus, the number of vanishing moments of the optimal mother wavelet should be between 3 and 5. An observation should be made in this regard. The first displacement modal shapes are considered to be the most accepted for damage detection since they are stable and easy to measure. However, new measure techniques an equipment is constantly developed. Rucka [32] used a scanning laser vibrometer and concluded that high vibration modes are the most sensitive to damage detection. Since the higher vibration modes require higher polynomials to approximate them, the number of vanishing modes should also increase. Douka et al. [19] stated that wavelet having high number of vanishing moments have more stable performance.

On the other hand, wavelet with more vanishing moments have longer support sizes, which induce wider duration of peaks. This effect worsens the crack location, in other words, localization worsens as support length increases. A compromise between the number of vanishing moments and support length has to be made.For all the reasons mentioned and after several proves, it was concluded that the optimal number of vanishing moment for multiresolution analysis is 4 and for damage detection is 2:Overall, rbio4.4 is chosen as the mother wavelet for SWT and rbio2.2 CWT (Figure 2).

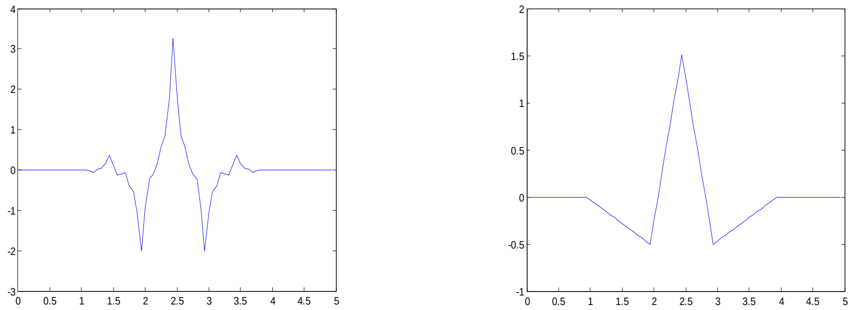

Figure 3-1. rbio2.2 wavelet: a) decomposition b) reconstruction

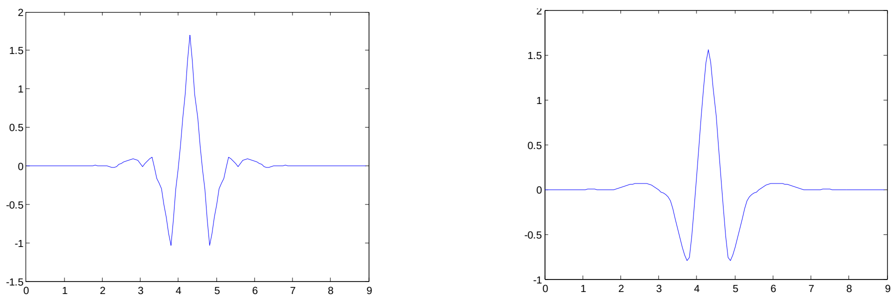

Figure 3. rbio4.4 wavelet: a) decomposition b) reconstruction

4 Combined Wavelet strategy for damage identification

In this section the proposed methodology for damage identification is described in four clear steps ones the data is already obtained. As all the vibration-based damage indicators requires information about the undamaged and damaged states of the structure. The data may be obtained in laboratory conditions or by field measurements if possible. An alternative, when experimental data is inaccessible, is performing numerical simulations and simulate environmental conditions adding noise to the response. For the proposed methodology, the method to obtain modal data is not important, thus not investigated in this work. The real contribution of this article is focused on how to deal with modal parameters i.e. modal shapes and natural frequencies ones they are already obtained.

Step 1: Extension of modal shapes

The Wavelet Transform is defined for an infinite interval. On the other hand, modal shapes signalsare spatial thus they are defined over a finite interval. When performing Wavelet Transform, this issue leads to wrong behavior at the beginning and at the end of the signal. There exist local changes in the mentioned points since the signal starts and finishes, however those discontinuities are not caused by damage. Damages close to the beginning andend of the structure can be masked due to the high values of the coefficients near the edges.

Different approaches to the problem have been proposed in the literature, like advanced mathematical methods in Messina and Gentile [33] or more simple ones like smooth extensions of the signal from beyond the boundaries (Rucka and Wilde[26], Solis et al. [2]). By Equation (2) it is possible to conclude that higher scales of CWT use more points to evaluate the signal, hence, thelength of the extension of the signal depends on the scales used in the transform.

When wavelet analysis is performed in the extended signal, the edge effect will appear at the end and beginning of the extended signal. The extended signal with boundary effects is going to be reshaped after Wavelet Transform is performed, leaving the original signal free of edge effects. In this article and after several verifications, a cubic linear extrapolation based on four neighboring points is going to be applied in both ends of the input signal. Each mode shape was also normalized to 1.

Step 2: Refinement and difference of modal shapes

The combination of SWT and CWT proposed by Cao and Quiao [1] consists of mode shape refinement by taking advantage of multiresolution analysis feature from SWT and then damage revealment using CWT. The refined modal shapes are the approximate coefficients resulting fromanalyzing the input data with SWT. If the decomposition level performed by SWT is too elevated, there is the risk that the discontinuity in modal shapes provoked by a damage may disappear. Moreover, it depends on the level of noise presented on the signal. In this work, a two-level decomposition with SWT will be performed to input modal shapes to refine them and obtain information related to the position of the damage.

As already stated, the methodology proposed by Cao and Quiao [1]does only applies to the first mode shape, which is considered as insufficient for damage detection since the sensibility of a mode shape to a punctual damage depends on its position. Solís et al. [2]proposed methodology was chosen to deal with this issue. They proposed a detection method based on the addition of the coefficients of the Continuous Wavelet Transform of the difference of modal shapes between undamaged and damaged states of the structure. Firstly,the difference of the extended refined modal shapes is then computed:

$$\Phi_{dif}(x) = \Phi_{dam}(x) - \Phi_{undam}(x) \qquad (17)$$

After the difference of modal shapes is obtained, Wavelet Transform of the difference of modal shapes is performed. The previous step for the *i*th mode shape can be notated as:

$$CWT^{i}(a,b) = \int_{-\infty}^{+\infty} \Phi_{dif}^{i}(x) \frac{1}{\sqrt{a}} \psi^{*}\left(\frac{x-b}{a}\right) dx \qquad (18)$$

Step 3: Addition and weighting of mode shapes

Zhong and Oyadiji [34]proposed a damage parameter based on the addition of the results for all modal shapes. Radziensky et al. [35] employed the change in natural frequencies for each mode to define a damage probability function which was used to weight the wavelet coefficients along the beam for each mode shape, and results for each mode shape were added up. The weighting of the coefficient is used to emphasize the most sensitive mode shapes. It is assumed that when the change in natural frequency is high, the difference between modal shapes will also be important. This way the modal shapes that do not change their natural frequencies are almost not taken into account since they do not bring new information; what is more, they could become a source of noise.

$$CWT^{i}_{weighted}(a,b) = CWT^{i}(a,b) * (1 - \frac{w^{i}_{undam}}{w^{i}_{dam}})^2 \qquad (14)$$

Where w^{i}_{undam} and w^{i}_{dam} are natural frequencies of mode shape *i* for undamaged and damaged states, respectively. The final step is to combine the results or each mode shape by adding them up. The results is a global wavelet coefficient.

$$CWT_{global}(a,b) = \sum_{i=1}^{m} CWT^{i}_{weighted}(a,b) \qquad (14)$$

Where $m$represent the number of modes added up.

Step 4: Normalization and analysis of the results

The value of Continuous Wavelet Transform coefficients rises with the scale. This issue could lead to a misinterpretation when analyzing the coefficients for each scale in a scalogram since the values for small scales could become masked by those of higher scales. For this reason, the global Wavelet Transform coefficients are normalized to the unity for each scale to obtain clearer results:

$$[CWT_{norm}]_a = \frac{[CWT_{global}(a,b)]_a}{\max [CWT_{global}(a,b)]_a} \qquad (14)$$

The normalized version of Continuous Wavelet Transform coefficients is going to be analyzed to detect and localize damage in the structure. Based on the analysis previously made, any abrupt change in the coefficients will be interpreted as the effect of a structural damage.

5 Results

5.1 Numerical simulation

A 3D model built with COMSOL is going to be used to simulate the deflection of a cracked beam. The left end is fixed at the coordinates of the origin. The material properties used are: Young's modulus E=183.236MPa, Poisson ratio v= 0.33 and mass density ρ= 7870 kg m^{-3}. Geometrical properties of the beam are length L=1m, width W=0.0258m and height H=0,0053m. The crack crosses all the width of the beam. The width of the crack$W_c$is set to 0.1mm while the height of it H_c is a parameter.The finite element type employed to perform the mesh was 3D tetrahedron element provided by COMSOL software. The number of meshing points is controlled by the software and depends on the shape of the structure, thus it changes with the size of the crack. A high meshing density is applied near the crack area so that its location can be accurately assessed. (Figures3 and 4)

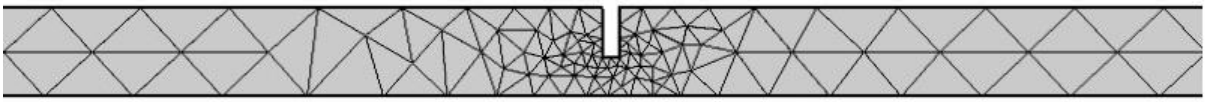

Figure 3. Finite element mesh of the beam.

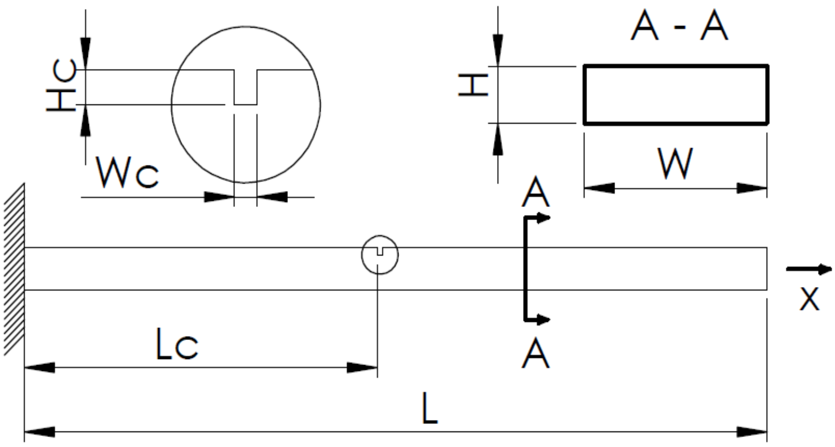

Figure 4. Clamped-free beam model.

The displacement of the nodes in the Y direction of the lower edge of the cantilever beam are selected to represent the deflection of the beam, thus to be the modal shapes. However, only the information of 10 equally-spaced nodes was obtained to simulate the position of the sensors along the beam.

The computing software MATLAB is selected for data processing in this project.Since damage indicators based on modal shapes require significant amount of points to accurately evaluate the state of the structure, an interpolation was performed between the values. Interpolation is also needed when measuring pointsare not equally spaced since they would create a non-uniform sampling input vector. In other words, damage indicators based on modal shapes must be applied to a vector with

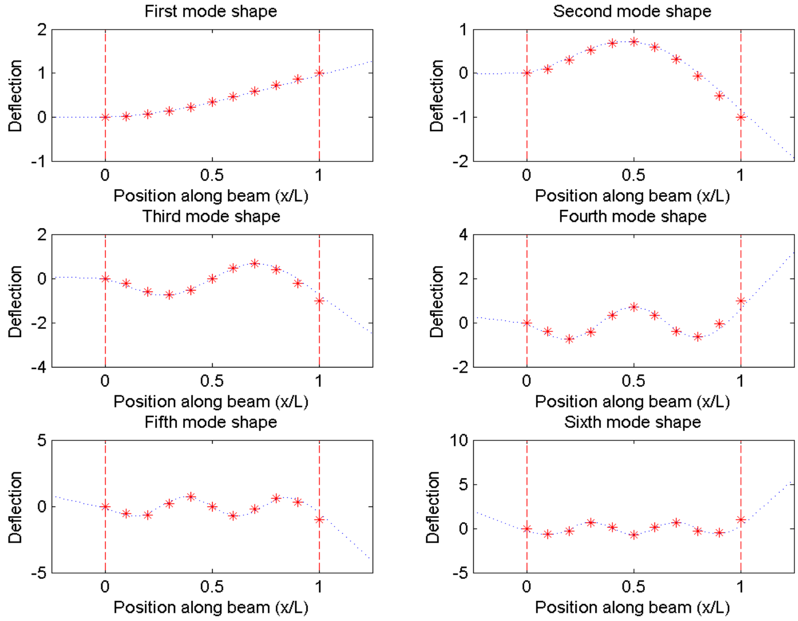

Figure 5.The stars correspond to the measured points (10 points) while the

dotted points are the interpolated modal shapes (128 points). Points beyond of

geometrically equally spaced components. If this was not the case, the measures could be interpolated to get a new vector which could feat that requirement. Another use of interpolation is to disregard measure points that can easily be detected as incorrect, whether due to noise influence or any external factor. It is also worth mentioning that there always should be measuring points at the end and the beginning of the beam. That way, the analyzed modal shapes are the result of the interpolation of the measuring points. Extrapolation, which could add uncertainties, is never done. In this article, a cubic linear interpolation was applied to obtain 128 interpolated points as shown in Figure 5.

In accordance with the modern literature, nine damage scenarios are chosen and simulated to analyze the behavior of damage indicators, where the position of the damage and its size are varied (Table 1).

Damage case	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Location (L_c) $[mm]$	100	100	100	500	500	500	900	900	900
Severity (H_c/H) [%]	1	10	25	1	10	25	1	10	25

Table 1. Damage cases studied.

5.2 Selection of modal shapes

.It is well known that the influence of a damage becomes almost imperceptible when it lies near the zero curvature zone of the mode shape.The effect of damages on modal shapes is in relation with the position of the modal nodes for each mode shape In other words, not all the modal shapes have the same sensitiveness to a particular damage. Furthermore,more important changes in natural frequencies are expected for the most sensitive modal shapes. To illustrate this effect, changes in natural frequencies and in Modal Assurance Criterion (MAC) inTable 2 are shown for Case 5.Concerning the changes in natural frequencies it is possible to conclude that modes 1, 2 and 4 are the most sensitive to the particular damage scenario. On the other hand, modes 3 and 5 almost does not show any change. This conclusion can be reinforced by analyzing MAC coefficients since they show the correlation between the analyzed mode shape and its undamaged version, being the value of 1 total correlation. Finally, a continuous wavelet analysis of each mode is presented in Figure 6.In this particular case (Case 5), modes 3 and 5 are not able correctly identify the position of the damage and they may introduce noise. In other words, damage detection and localization cannot rely in only one single mode shape.

Mode	**1**	**2**	**3**	**4**	**5**	**6**
Undamaged frequencies [Hz]	4.1311	25.8896	72.4918	142.055	234.827	350.791
Damaged frequencies [Hz]	4.12355	25.6877	72.4916	140.976	234.819	348.17
Relative difference of frequencies [%]	0.1846	0.77981	0.00025	0.75897	0.00320	0.746
MAC coefficient	0.99999	0.99993	0.99999	0.99958	0.99999	0.99881

Table 2. Frequencies and MAC coefficient for each modal shape for Case 5 ($L_c = 500mm, H_c/H = 10\%$) damage scenario.

Identification of the most sensitive mode shape

This scenario justifies the weighting of modal shapes in order to reduce those unwanted effects. On the other hand, mode 6 may be able to offer useful information, however 10 sensors is not considered

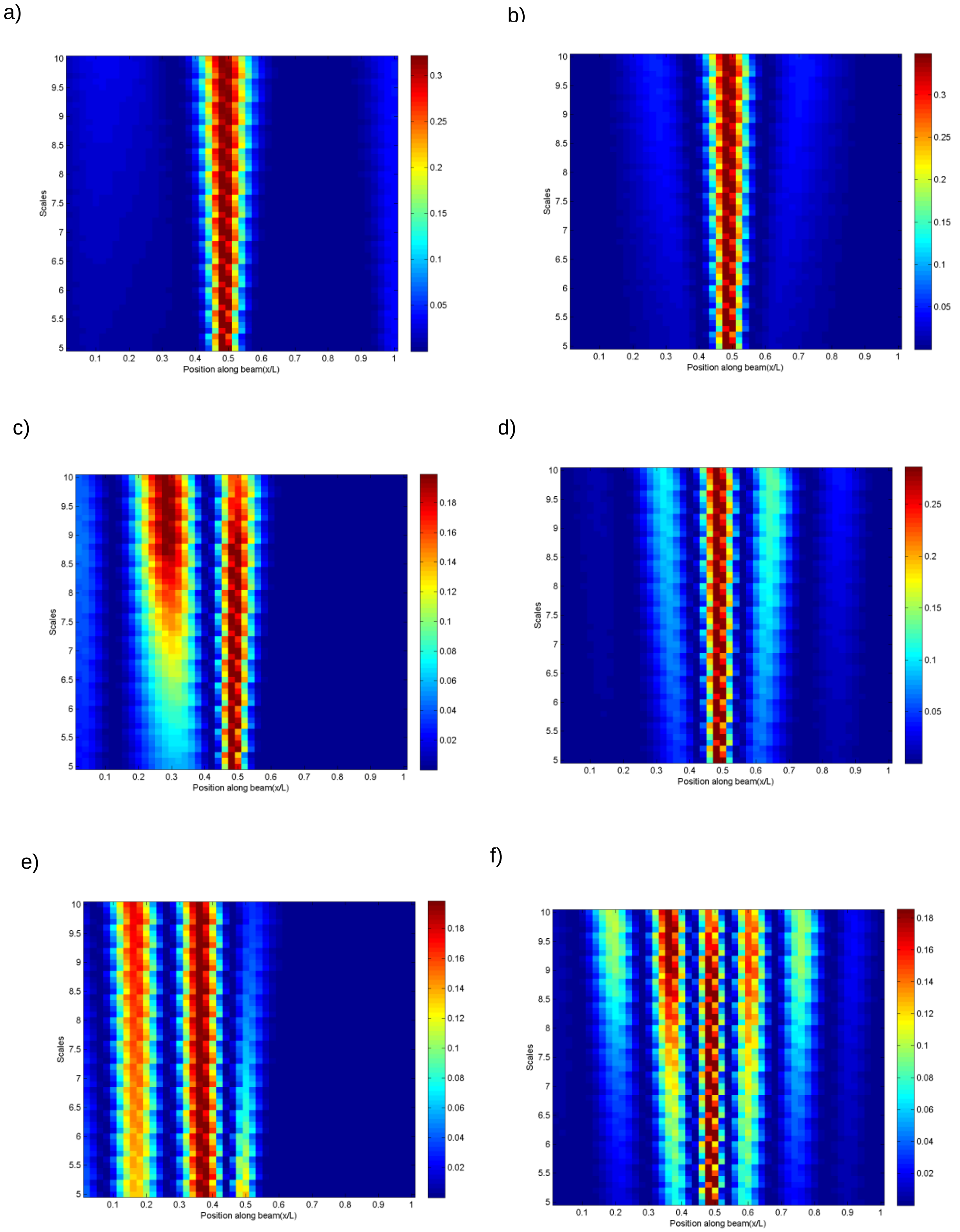

Figure 6. CWT coefficients for individual modal shapes for a particular scenario. (a) first modal shape, (b) second modal shape, (c) third modal shape, (d) fourth modal shape, (e) fifth modal shape, (f) sixth modal shape.

enough to correctly interpolate it. Therefore, for the rest of the paper there are analyzed the first 4 modes, since from a practical point of view, the lowest modal shapes are the most suitable to obtain.

It was previously mentioned that the position of the damage becomes almost imperceptible when it lies near the zero zone of a mode shape. In the particular case of the cantilever beam, all the modal shapes have a modal node at the beginning of the beam, therefore, the beginning of it is expected to be the most problematic zone of the beam.

5.3 Effects of crack location and size

The proposed methodology based on Wavelet Transform will be compared with some of the most known damage indicators in the bibliography: Strain Energy Damage Indicator (SEDI), Damage Index Indicator (DI) and Modal Shape Curvature (MSC). Figures 7-15 show the behavior of damage indicators for all damage cases, which are described in Table 1. Figure 7, 8 and 9 confirms the hypotesis already made that the beginning of the beam is aproblematic zone to detect and localize damage. In Figure 7-a it can be objected that SEDI completely fails to locate the damage when it is small and close to the beginning of the beam. On the other hand, the maximum absolute coefficients of DI and MSC does coincide with the actual position of fissure, however the influence of computational noise is remarkable; on a practical situation, a smaller damage may be hidden in those situations. Figures 8 and 9 shows that this effects are reduced when the damage size is increased since the difference between the high values produced by the damage and those provoked by noise increases. In the case of Wavelet Analysis, damage’s location is correctly assessed by the modulus maximum line and does not give doubts about the actual position of the damage.Figures 10-12 show the behavior of damage indicator when the damage is in the middle of the beam. The maximum absolute coefficients of SEDI, DI and MSC do coincide with the actual position of the damage for the three cases, the estimation on the location of the damage is exact. On the other hand, the location of these damage indicators is not precise since uncertain zones are presented near the maximum coefficients. In Figure 10, the peaks on both sides of the maximum coefficients may be easily confounded with two independent damages. This effect is attenuated when the size of the damage increases. In the case of Wavelet Analysis, this problem is partly solved by weighting the modal shapes that may introduce noise to the final result.

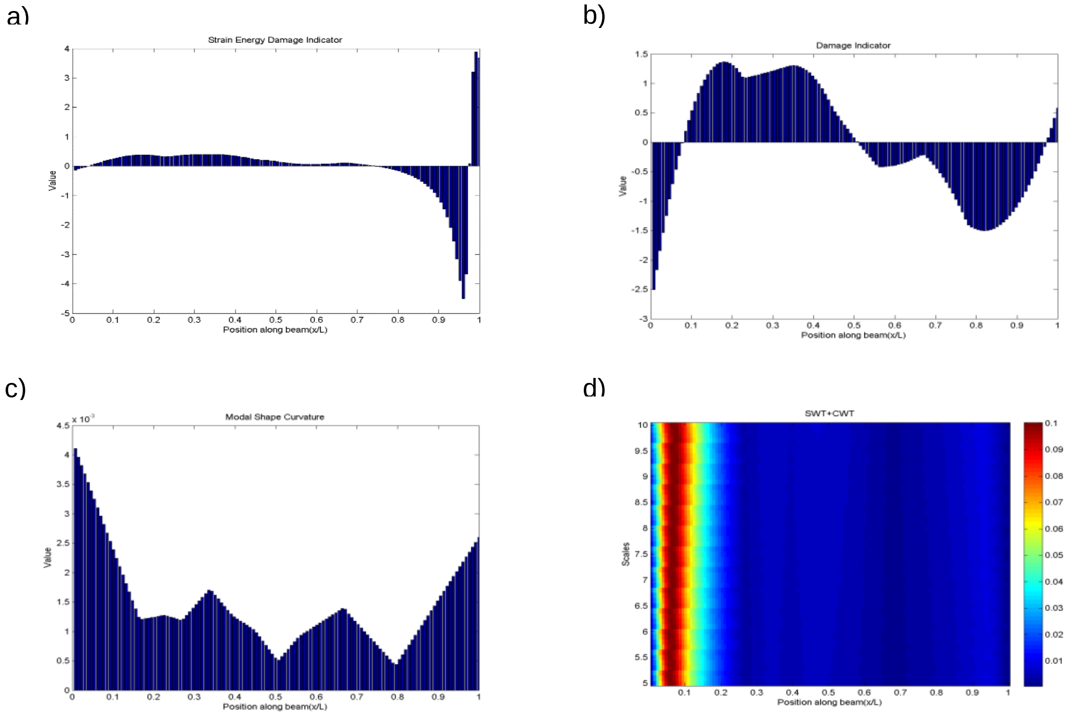

Figure 7. Comparison of damage indicators for Case 1: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet

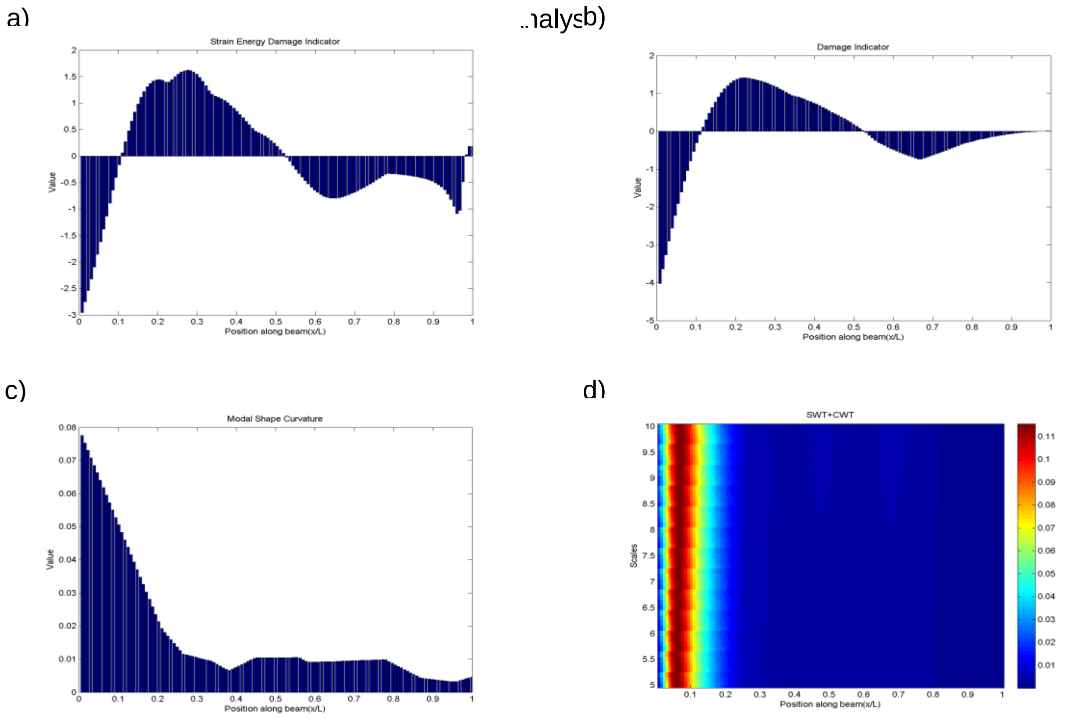

Figure 8. Comparison of damage indicators for Case 2: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

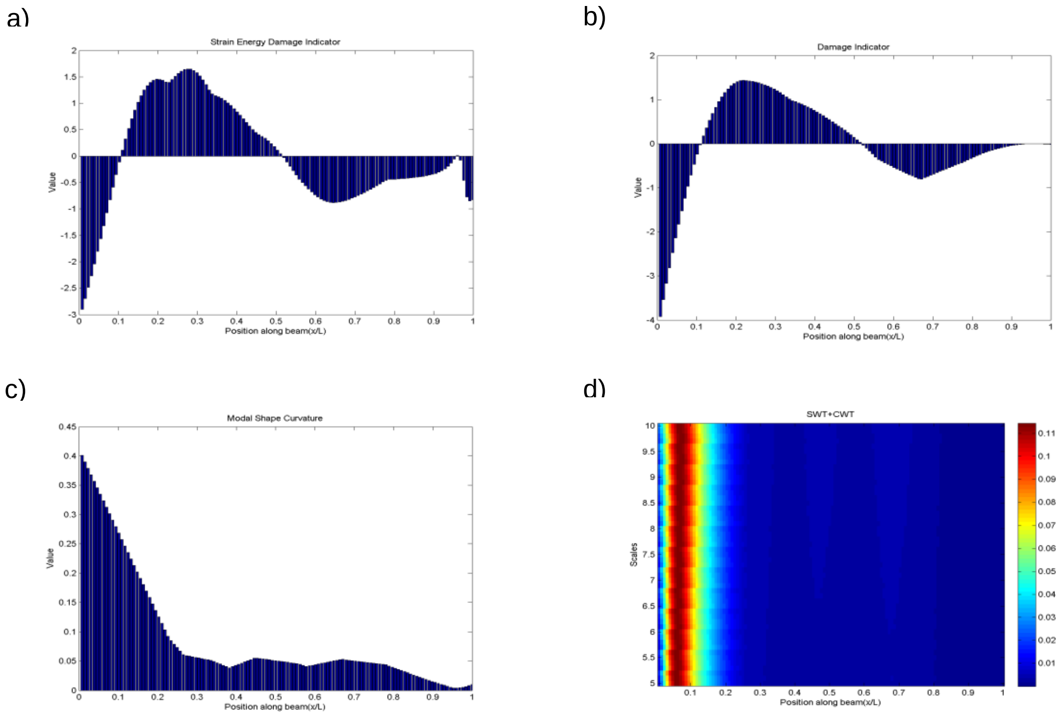

Figure 9. Comparison of damage indicators for Case 3: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet

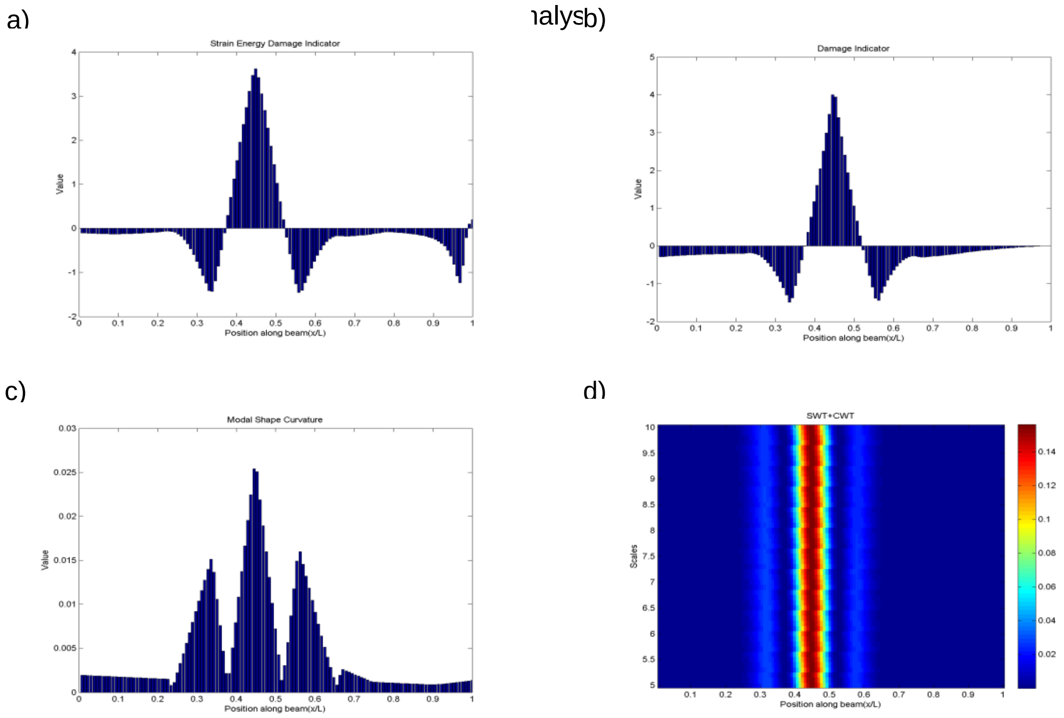

Figure 10. Comparison of damage indicators for Case 4: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

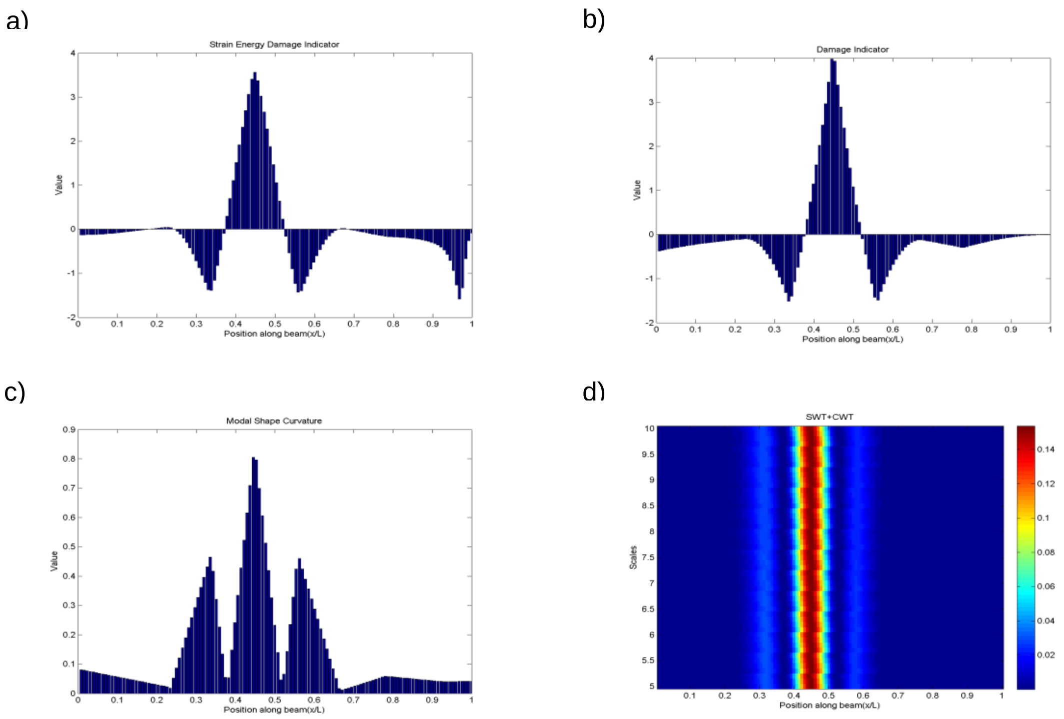

Figure 11. Comparison of damage indicators for Case 5: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

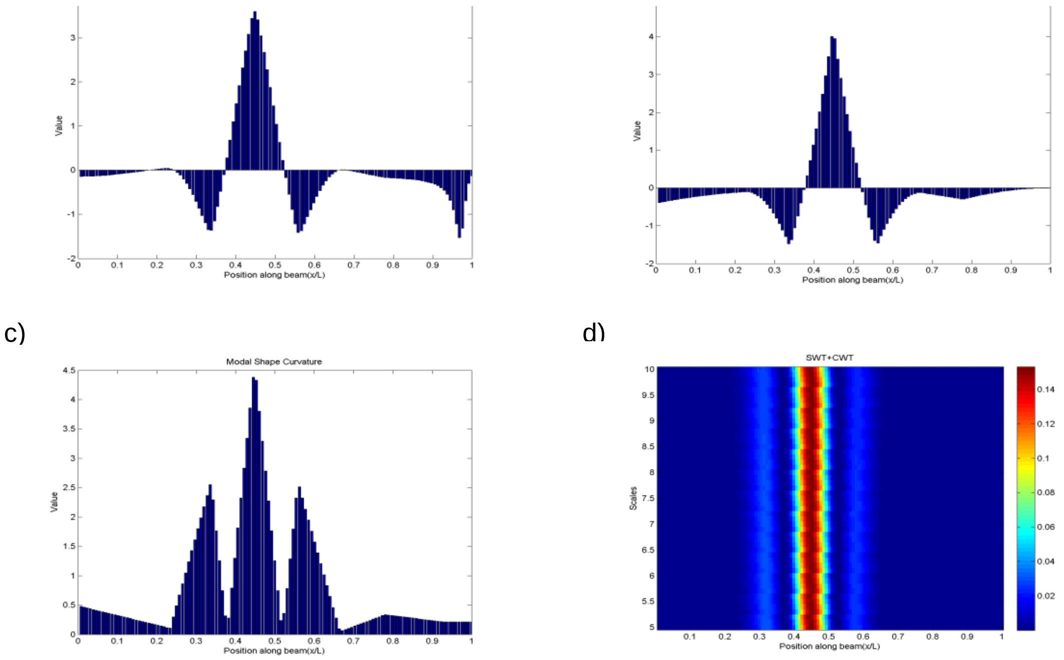

Figure 12. Comparison of damage indicators for Case 6: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

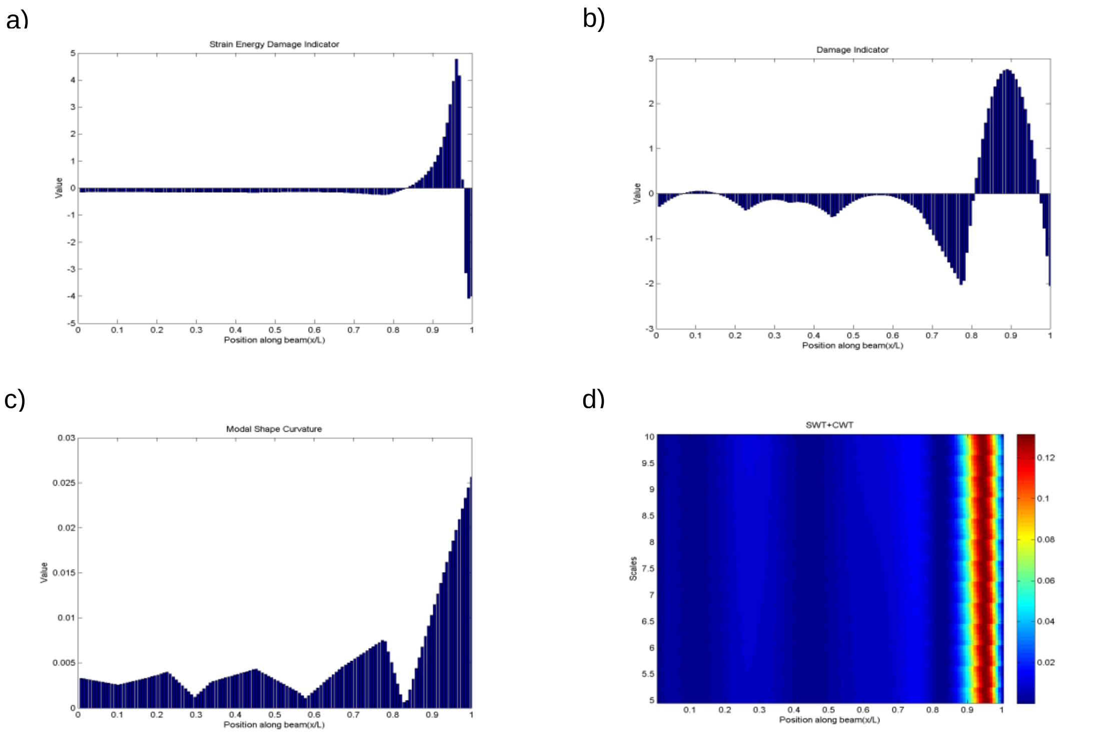

Figure 13. Comparison of damage indicators for Case 7: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet nalysi

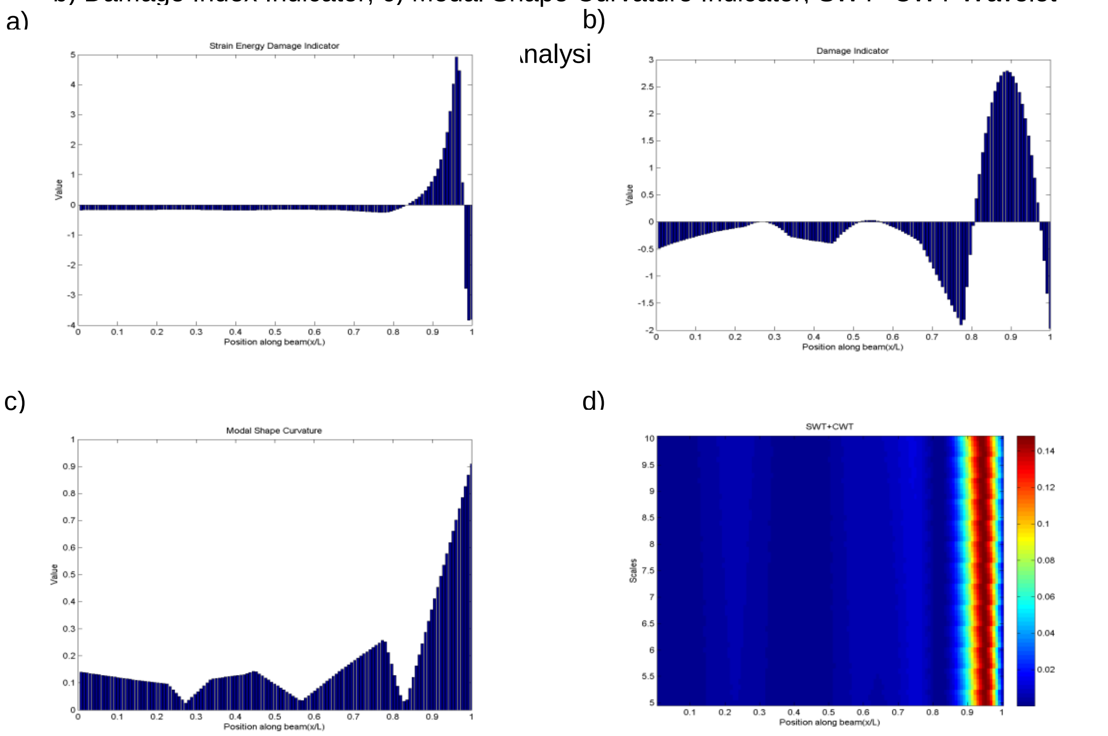

Figure 14. Comparison of damage indicators for Case 8: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

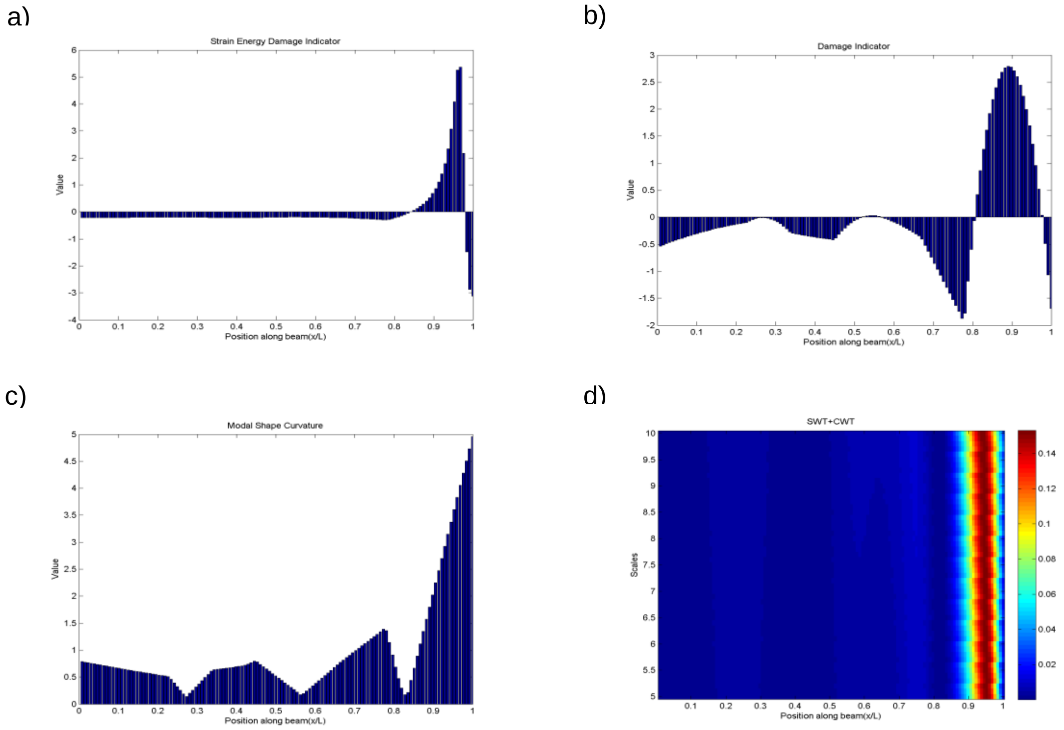

Figure 15. Comparison of damage indicators for Case 9: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

Figures

13-15 illustrate that damages in the end zone of the beam are the most easily located. Neither of the indicators have problems with detecting nor correctly assessing the location of the damage Moreover, the computational noise is less compared to the one in the beginning and middle cases. Wavelet Analysis has also the most solid performance in these cases since the computational noise is almost imperceptible.

5.4 Accuracy of damage location

To compare the accuracy of damage location, the position of the maximum absolute coefficients for each damage indicator is considered as its estimation to damage’s position (Table 3). In the case of CWT, a multiscale analysis is performed which gives different coefficients for each scale. By Equation 2 it can be deduced that big scales are prone to general characteristics of the signal while small scales are more sensitive to local events. Since damages provoke a local discontinuity in modal shapes, scale 5 is chosen in this paper to evaluate the location accuracy of wavelet methodology. From Figures 7-15 it can be concluded that the damage accuracy of the analyzed damage indicators does not depend on the size of the damage. The location accuracy of damage indicators should be analyzed with respect to the spatial distance between the simulated sensors. In the proposed case, the corresponded distance is 0.1m, therefore a threshold of 0.1m to both sides of the certain location of the damage can be defined.

Position of the damage	SEDI	DI	MSC	Wavelet
$\mathbf{L_c = 0.1}\boldsymbol{m}$	0.2734375	0.21875	0.0078125	0.046875
$\mathbf{L_c = 0.5}\boldsymbol{m}$	0.4453125	0.4453125	0.4453125	0.4453125
$\mathbf{L_c = 0.9}\boldsymbol{m}$	0.9609375	0.890625	1	0.9609375

Table 3. Position of maximum coefficient for each damage indicator vs position of the fissure.

The spatial distance between sensors can be taken into account with the following formula:

$$\mathrm{L}_{relative} = \left|\frac{\mathrm{L}_{result} - \mathrm{L}_c}{\mathrm{D}_{sensors}}\right| \qquad (14)$$

Where $\mathrm{D}_{sensors}$ is the spatial distance between simulated sensors, L_c is the real positon of the damage and L_{result} is the damage location estimated by damage indicators.

If the indicator result is within the threshold zone, the estimation of damage position can be assessed as correct. In terms of Table 4, a good assessment of damage location is indicated with coefficients lesser than 0.1.When the damage was located close to the beginning of the beam, only Wavelet

Position of the damage	SEDI	DI	MSC	Wavelet
$\mathbf{L_c = 0.1}\boldsymbol{m}$	0.1734375	0.11875	0.0921875	0.053125
$\mathbf{L_c = 0.5}\boldsymbol{m}$	0.0546875	0.0546875	0.0546875	0.0546875

$L_c = 0.9m$	0.0609375	0.009375	0.1	0.0609375

Table 4. Relative position of maximum coefficient for each damage indicator vs position of the fissure, regarding the spatial distance between simulated sensors (0.1m).

Analysis and MSC were able to assess its position precisely. In the middle case, all the indicators proved to have the same level of performance. In the last case, Wavelet Analysis and SEDI gave good enough results while the assessment of DI was almost perfect. On the other hand, MSC fails in this case. Overall, it can be concluded that the methodology based on Wavelet Analysis is the most trustworthy in terms of accuracy of damage detection since in all the cases the assessed position of the damage position was close enough to the real position of it.

5.5 Influence of noise

In order to evaluate the detectability of damage indicators in real measurement conditions, noise was added to the simulated numerical data. To the modal shapes obtained by finite element analysis it was added normal distributed white noise. The measurement of noise intensity added, defined as the power of the desired signal to the noise power, was SNR= 65%.

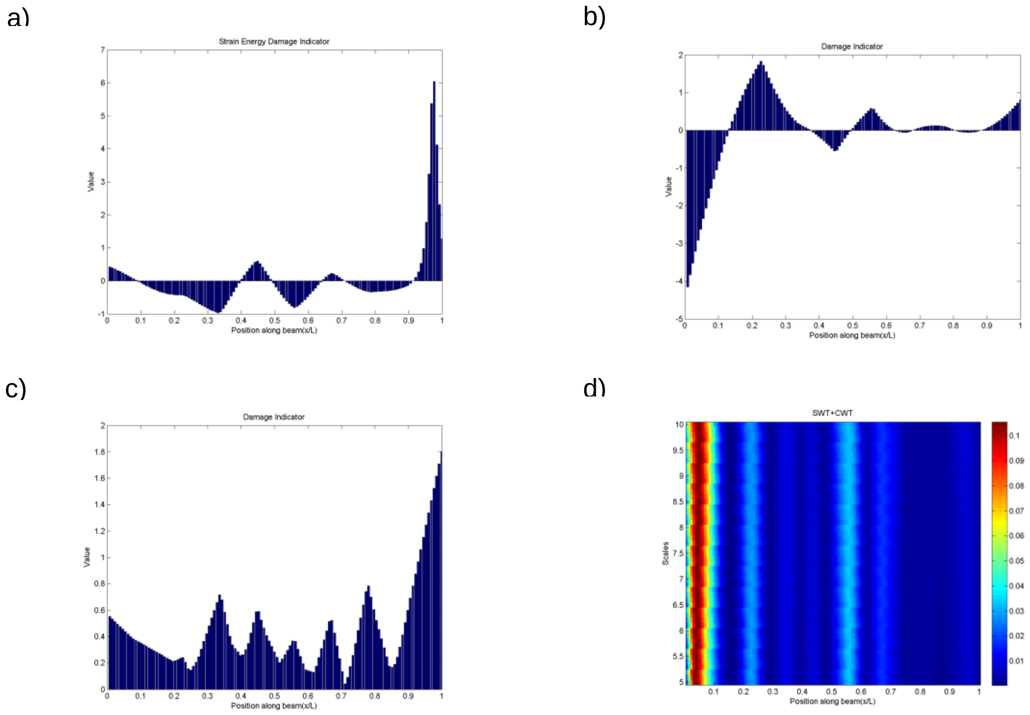

Figure 17. Comparison of damage indicators for Case 3 with noisy signals: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

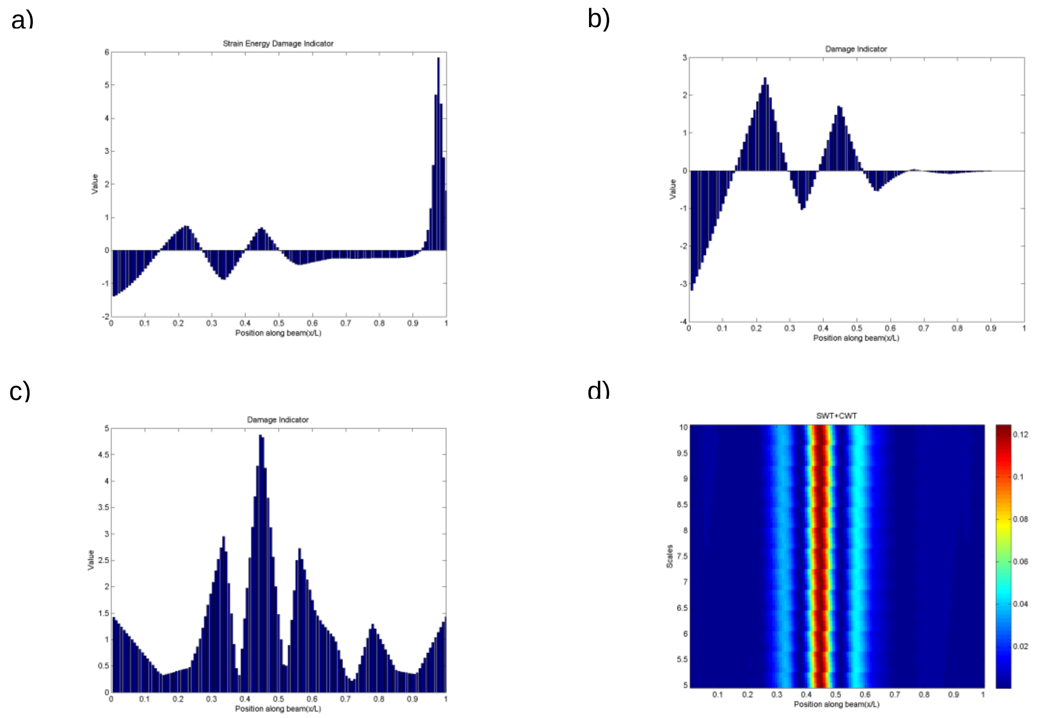

Figure 18. Comparison of damage indicators for Case 6 with noisy signals: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, ...NT Wavelet

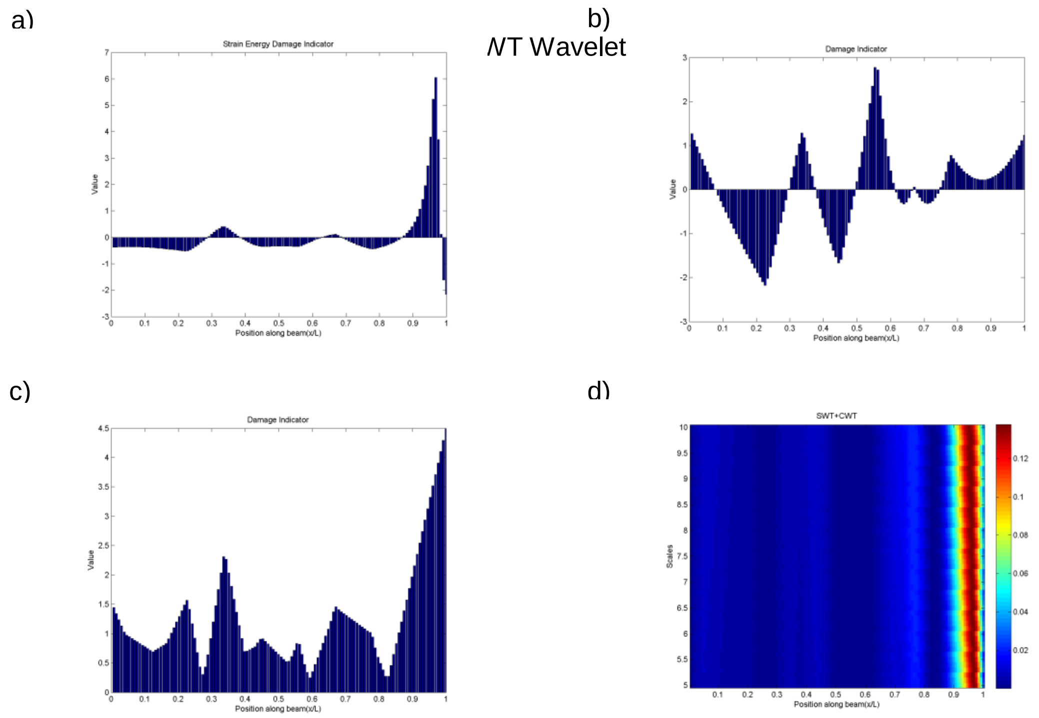

Figure 19. Comparison of damage indicators for Case 9 with noisy signals: a) Strain Energy Damage Indicator, b) Damage Index Indicator, c) Modal Shape Curvature Indicator, SWT+CWT Wavelet Analysis

Crack localization is easier in case of large damage since if it was small, it can be easily confounded with noise. As a matter of fact, Figures 17-19 shows the behavior of damage indicators to noisy signals for Cases 3, 6 and 9. Clearly Wavelet Analysis outperforms classical indicators in terms of noisy signals, giving accurate and precise assessment of damage location in all the cases. On the other hand, classical indicators were not able to cope with the challenge.

A multiscale analysis using CWT can also be performed to identify the damage position in a noisy signal. It is well known that wavelet coefficients increase with the scale in presence of a discontinuity. Figure 20 is a 3D plot of the wavelet coefficients without normalization of Case 6. Local maximum can be detected in different positions along the beam. However, those coefficients does not decrease with scale smoothly; on the other hand, they tend to be constant. This behavior can be used as a tool to differentiate the effect of the damage to the effect of the noise by looking at the decay behavior of wavelet coefficients. It can be seen that at $x = 0.5m$ the decay behavior is more important than in the rest of the beam, which indicates the presence of a damage.

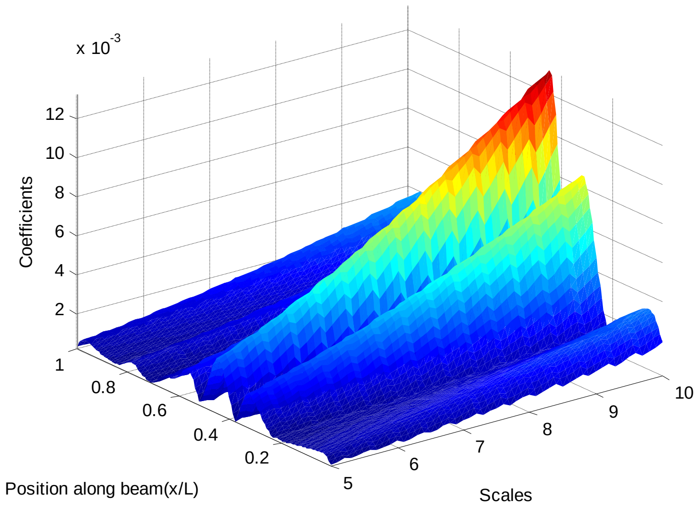

Figure 20. Three-dimensional plot of wavelet coefficients for noisy data, Case 5

6 Conclusions

A methodology for identifying damage in structures was proposed.The fundamental modal shapes of a potentially damaged structure are refined with SWT multiresolution analysis to separete noise and intererences componentes from damage-related information.The differences between the refined modal shapes and those corresponding to the saine state of the structure are afterwards computed. CWT is applied, due to its ability to perform multiscale analysis, to the difference of mode shapes. The resulting coefficients are weighted depending on the changes in natural frequencies to reduce the potential sources of noise. Finally, the coefficients are added up and normalized to obtain a single result which enables fast detection and location of the damage.

A comparison between the proposed methodology and well-known classical indicators was carried out. The results proved that the proposed algorithm gives better results than classical indicators with regard to accurate location of damage. It also showed indiference to the size of damage, showing sensitivity to little damages. Furthermore, the proposed methodology outperforms the previous indicators capacity of detecting and locating damages in the presence of noise.

In the second part of this work, the methodology will be tested with experimental data in order to validate its efficiency in real cases.

References

[1] M. Cao and P. Qiao, "Integrated wavelet transform and its application to vibration mode shapes for the damage detection of beam-type structures," Smart Materials and Structures, vol. 17, no. 5, p. 055014, Aug. 2008..

[2] M. Algaba, M. Solis, and P. Galvan, "A combined wavelet-modal analysis for damage location in beams," Proc. Int. Conf. Noise Vib. Eng. ISMA 2012, pp. 777–790, 2013.

[3] P. Cawley and R. D. Adams, "The location of defects in structures from measurements of natural frequencies," The Journal of Strain Analysis for Engineering Design, vol. 14, no. 2, pp. 49–57, Apr. 1979.9.

[4] O. S. Salawu, "Detection of structural damage through changes in frequency: a review," Engineering Structures, vol. 19, no. 9, pp. 718–723, Sep. 1997.

[5] R. Serra, and J.-M. Mencik G. Gautier, "A Subspace Fitting Method based on Finite Elements for Identification and Localisation of Damage in Mechanical Systems," Proceedings of the Eleventh International Conference on Computational Structures Technology.

[6] D. L. Brown and R. J. Allemagn, "A correlation coefficient for modal vector analysis," First International Modal Analysis Conference, pp. 110-116, 1982.

[7] J.-T. Kim, Y.-S. Ryu, H.-M. Cho, and N. Stubbs, "Damage identification in beam-type structures: Frequency-based method vs mode-shape-based method," Engineering Structures, vol. 25, no. 1, pp. 57–67, Jan. 2003.

[8] N. A. J Lieven and D. J. Ewins, "Spatial Correlation of Mode Shapes: the Coordinate Modal Assurance Criterion (COMAC)," Proceedings, International Modal Analysis Conference, pp. 690-695, 1988.
[9] A. K. Pandey, M. Biswas, and M. M. Samman, "Damage detection from changes in curvature mode shapes," Journal of Sound and Vibration, vol. 145, no. 2, pp. 321–332, Mar. 1991.
[10] M. De Roeck and G. A. Wahab, "Damage Detection in Bridges Using Modal Curvatures: Application To a Real Damage Scenario," 1999.
[11] O. S. Salawu and C. Williams, "Damage location using vibration mode shapes," in Proceedings of the 12th International Modal Analysis Conference, pp. 933–939, Honolulu, Hawaii, USA, 1994..
[12] J. T. Kim, and C. R. Farrar N. Stubbs, "Field Verification of a nondestructive damage localization and severity estimation algorithm," Proceedings of the 13th International Modal Analysis Conference (IMAC XIII), vol. 182, pp. 210-218, 1995.
[13] E. Sazonov and P. Klinkhachorn, "Optimal spatial sampling interval for damage detection by curvature or strain energy mode shapes," Journal of Sound and Vibration, vol. 285, no. 4-5, pp. 783–801, Aug. 2005.
[14] A. K. Pandey and M. Biswas, "Damage detection in structures using changes in flexibility," Journal of Sound and Vibration, vol. 169, no. 1, pp. 3–17, Jan. 1994.
[15] I. Daubechies, "Ten lectures on Wavelets," The Journal of the Acoustical Society of America, vol. 93, no. 3, p. 1671, 1993..
[16] C. Surace and R. Ruotolo,."Crack detection of a beam using the wavelet transform". The International Society for Optical Engineering, 1994.
[17] K. Liew, and Q. Wang, "Application of Wavelet Theory for Crack Identification in Structures." J. Eng. Mech., 152-157, 1998
[18] J. Hong, Y. Y. Kim, H. C. Lee, and Y. W. Lee, "Damage detection using the Lipschitz exponent estimated by the wavelet transform: Applications to vibration modes of a beam," International Journal of Solids and Structures, vol. 39, no. 7, pp. 1803–1816, Apr. 2002.
[19] E. Douka, S. Loutridis, and A. Trochidis, "Crack identification in beams using wavelet analysis," International Journal of Solids and Structures, vol. 40, no. 13-14, pp. 3557–3569, Jun. 2003.
[20] B. Chen, Y. Kang, P. Li, and W. Xie, "Detection on structural sudden damage using continuous Wavelet transform and Lipschitz Exponent," Shock and Vibration, vol. 2015, pp. 1–17, 2015.
[21] N. Wu and Q. Wang, "Experimental studies on damage detection of beam structures with wavelet transform," International Journal of Engineering Science, vol. 49, no. 3, pp. 253–261, Mar. 2011.
[22] G., Venini, P. Naldi, "Wavelet analysis of structures: statics, dynamics and damage identification.", Meccanica, pp :223-230, 1997.
[23] C.-J. Lu and Y.-T. Hsu, "Vibration analysis of an inhomogeneous string for damage detection by wavelet transform," International Journal of Mechanical Sciences, vol. 44, no. 4, pp. 745–754, Apr. 2002.
[24] S. G. Mallat, "A theory for multiresolution signal decomposition: The wavelet representation," IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 11, no. 7, pp. 674–693, Jul. 1989..
[25] S. Zhong and S. O. Oyadiji, "Crack detection in simply supported beams without baseline modal parameters by stationary wavelet transform," Mechanical Systems and Signal Processing, vol. 21, no. 4, pp. 1853–1884, May 2007..
[26] M. Rucka and K. Wilde, "Application of continuous wavelet transform in vibration based damage detection method for beams and plates," Journal of Sound and Vibration, vol. 297, no. 3-5, pp. 536–550, Nov. 2006..
[27] M. Masoumi and M. R. Ashory, "Damage identification from uniform load surface using continuous and stationary wavelet transforms," Latin American Journal of Solids and Structures, vol. 11, no. 5, pp. 738–754, Oct. 2014.
[28] S. G. Mallat, A wavelet tour of signal processing, 2nd ed. San Diego, CA: Academic Press, 1999..

[29] A. V. Ovanesova and L. E. Suárez, "Applications of wavelet transforms to damage detection in frame structures," Engineering Structures, vol. 26, no. 1, pp. 39–49, Jan. 2004.
[30] B. K. R. Prasad, N. Lakshmanan, K. Muthumani, and N. Gopalakrishnan, "Enhancement of damage indicators in wavelet and curvature analysis," Sadhana, vol. 31, no. 4, pp. 463–486, Aug. 2006.
[31] X. Jiang, Z. J. Ma, and W.-X. Ren, "Crack detection from the slope of the mode shape using complex continuous Wavelet transform," Computer-Aided Civil and Infrastructure Engineering, vol. 27, no. 3, pp. 187–201, Sep. 2011.
[32] M. Rucka, "Damage detection in beams using Wavelet Transform on higher vibration modes," Journal of Theorical and Applied Mechanics, vol 49, no 2, pp. 399-417, 2011.
[33] A. Gentile and A. Messina, "On the continuous wavelet transforms applied to discrete vibrational data for detecting open cracks in damaged beams," International Journal of Solids and Structures, vol. 40, no. 2, pp. 295–315, Jan. 2003.
[34] S. Zhong and S. O. Oyadiji, "Detection of cracks in simply-supported beams by continuous wavelet transform of reconstructed modal data," Computers & Structures, vol. 89, no. 1-2, pp. 127–148, Jan. 2011.
[35] M. Radzieński, M. Krawczuk, and M. Palacz, "Improvement of damage detection methods based on experimental modal parameters," Mechanical Systems and Signal Processing, vol. 25, no. 6, pp. 2169–2190, Aug. 2011.